

Global Paths of Time-Periodic Solutions of the Benjamin-Ono Equation Connecting Arbitrary Traveling Waves

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November 26, 2008

Abstract

We classify all bifurcations from traveling waves to non-trivial time-periodic solutions of the Benjamin-Ono equation that are predicted by linearization. We use a spectrally accurate numerical continuation method to study several paths of non-trivial solutions beyond the realm of linear theory. These paths are found to either re-connect with a different traveling wave or to blow up. In the latter case, as the bifurcation parameter approaches a critical value, the amplitude of the initial condition grows without bound and the period approaches zero. We propose a conjecture that gives the mapping from one bifurcation to its counterpart on the other side of the path of non-trivial solutions. By experimentation with data fitting, we identify the form of the exact solutions on the path connecting two traveling waves, which represents the Fourier coefficients of the solution as power sums of a finite number of particle positions whose elementary symmetric functions execute simple orbits in the complex plane (circles or epicycles). We then solve a system of algebraic equations to express the unknown constants in the new representation in terms of the mean, a spatial phase, a temporal phase, four integers (enumerating the bifurcation at each end of the path) and one additional bifurcation parameter. We also find examples of interior bifurcations from these paths of already non-trivial solutions, but we do not attempt to analyze their algebraic structure.

Key words. Periodic solutions, Benjamin-Ono equation, non-linear waves, solitons, bifurcation, continuation, exact solution, adjoint equation, spectral method

AMS subject classifications. 65K10, 37M20, 35Q53, 37G15

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1 Introduction

The Benjamin-Ono equation is a non-local, non-linear dispersive equation intended to describe the propagation of internal waves in a deep, stratified fluid [4, 12, 25]. In spite of non-locality, it is an integrable system with meromorphic particle solutions [10], N -soliton solutions [20], and N -phase multi-periodic solutions [27, 13, 22]. A bilinear formalism [27] and a Bäcklund transformation [23, 5, 21] have been found to generate special solutions of the equation, and, in the non-periodic setting of rapidly decaying initial conditions, an inverse scattering transform has been developed [14, 17] that exploits an interesting Lax pair structure in which the solution plays the role of a compatibility condition in a Riemann-Hilbert problem.

It is common practice in numerical analysis to test a numerical method using a problem for which exact solutions can be found. Our initial interest in Benjamin-Ono was to serve as such a test problem. Although many of the tools mentioned above can be used to study time-periodic solutions, they do not generalize to problems such as the vortex sheet with surface tension [1] or the true water wave [26, 16], which are not known to be integrable. Our goal in this paper is to develop tools that *will* generalize to these harder problems and use them to study bifurcation and global reconnection in the space of time-periodic solutions of B-O. Specifically, we employ a variant of the numerical continuation method we introduced in [2] for this purpose, which yields solutions that are accurate enough that we are able to recognize their analytic form.

Because we approached the problem from a completely different viewpoint, our description of these exact solutions is very different from previously known representations of multi-periodic solutions. Rather than solve a system of non-linear algebraic equations at each x to find $u(x, t)$ as was done in [22], we represent $u(x, t)$ in terms of its Fourier coefficients $c_k(t)$, which turn out to be power sums $c_k = 2[\beta_1^k + \cdots + \beta_N^k]$ of a collection of N particles $\beta_j(t)$ evolving in the unit disk of the complex plane as the zeros of a polynomial $z \mapsto P(z, t)$ whose coefficients execute simple orbits (circles or epicycles in \mathbb{C}). The connection between the new representation and previous representations will be explored in [31].

Many of our findings on the structure of bifurcations and reconnections in the manifold of time-periodic solutions of the Benjamin-Ono equation are likely to hold for many other systems as well. One interesting pitfall we have identified by applying our method to an integrable problem is that degenerate bifurcations can exist that are not predicted by a linearization about traveling waves. Although it is possible that such degeneracy is a consequence of the symmetries that make this problem integrable, it seems likely that other

problems such as the water wave will also possess degenerate bifurcations that are invisible to a linearized analysis. We have also found that one cannot achieve a complete understanding of these manifolds of time-periodic solutions by holding e.g. the mean constant and varying only one parameter. In some cases, varying this parameter leads to a re-connection with another traveling wave, while in other cases the solution (i.e. the L^2 norm of the initial condition) blows up as the parameter approaches a critical value. Only by varying both the mean and this parameter simultaneously (and accounting for spatial and temporal phase shifts) does a correct picture emerge in which every (three-parameter) family of traveling waves is embedded as a submanifold of a four-parameter family of time-periodic solutions that connects it to a different three-parameter family of traveling waves. Thus, although numerical continuation with more than one parameter is difficult, it will likely be necessary to explore multi-dimensional parameter spaces to achieve a thorough understanding of time-periodic solutions of other problems.

On the numerical side, we believe our use of certain Fourier modes of the initial conditions as bifurcation parameters will prove useful in many other problems beyond Benjamin-Ono. We also wish to advocate the use of variational calculus and optimal control for the purpose of finding time-periodic (or otherwise special) solutions of non-linear PDE. Although variational methods have been used extensively to find periodic solutions of ODE such as the Lorenz attractor [28], there have been very few studies of time-periodic solutions of PDE that employ these tools; see e.g. [6, 15] for notable exceptions. Many authors do not attempt to find exact periodic solutions, and instead point out that typical solutions of certain equations do tend to pass near their initial states at a later time; see e.g. [9]. If true periodic solutions are sought, a more common approach has been to either iterate on a Poincaré map and use stability of the orbit to find time-periodic solutions [8], or to mimic the ODE methods and define a Newton iteration to find a fixed point of the Poincaré map using finite differences for the Jacobian [29]. We have found that it is much more efficient and accurate to define a non-linear functional measuring deviation from periodicity, and to drive this functional to zero using a quadratically convergent quasi-Newton method in which the first variation of the functional is computed by solving an adjoint PDE.

This paper is organized as follows. In Section 2, we discuss stationary, traveling and particle solutions of B-O, linearize about traveling waves, and classify all bifurcations predicted by linear theory from traveling waves to non-trivial time-periodic solutions. In Section 3, we present a collection of numerical experiments using our continuation method to follow several paths of non-trivial solutions beyond the realm of linear theory in order to formulate a conjecture that gives the global mapping from one traveling wave bifurcation to its counterpart on the other side of the path. In Section 4, we study the behavior of the Fourier

modes of the time-periodic solutions found in Section 3 and propose a conjecture about the exact form of these solutions. We take the first steps toward proving this conjecture by deriving algebraic relationships between the coefficients of the ansatz in order to explain why some paths of solutions reconnect and others lead to blow-up. We leave a full proof of these conjectures to a later paper in which the connection between our exact solutions and previously known, multiperiodic solutions is explored [31]. Finally, in Section 5, we discuss interior bifurcations from these paths of already non-trivial solutions to still more complicated solutions. Although the existence of such a hierarchy of solutions was already known [27], bifurcation between various levels of the hierarchy has not previously been discussed, and has interesting implications on the limitations of linearization for the prediction of bifurcations.

2 Bifurcation from Traveling Waves

In this section, we study the linearization of the Benjamin-Ono equation about stationary solutions and traveling waves by solving an infinite dimensional eigenvalue problem in closed form. Each eigenvector corresponds to a time-periodic solution of the linearized equation. The traveling case is reduced to the stationary case by requiring that the period of the perturbation (with a suitable spatial phase shift) coincide with the period of the traveling wave. The main goal of this section is to devise a classification scheme of the bifurcations from traveling waves so that in later sections we can describe which (local) bifurcations are connected together by a global path of non-trivial time-periodic solutions.

2.1 Stationary, Traveling and Particle Solutions

We consider the Benjamin-Ono equation on the periodic interval $\mathbb{R}/2\pi\mathbb{Z}$, namely

$$u_t = Hu_{xx} - uu_x. \quad (1)$$

Here H is the Hilbert transform, which has the symbol $\hat{H}(k) = -i \operatorname{sgn}(k)$. This equation possesses solutions [10, 2] of the form

$$u(x, t) = \alpha_0 + \sum_{l=1}^N u_{\beta_l(t)}(x), \quad (2)$$

where α_0 is the mean, $\beta_1(t), \dots, \beta_N(t)$ are the trajectories of N particles evolving in the unit disk Δ of the complex plane and governed by the ODE

$$\dot{\beta}_l = \sum_{\substack{m=1 \\ m \neq l}}^N \frac{2i}{\beta_l^{-1} - \beta_m^{-1}} + \sum_{m=1}^N \frac{2i\beta_l^2}{\beta_l - \beta_m^{-1}} + i(1 - \alpha_0)\beta_l, \quad (1 \leq l \leq N), \quad (3)$$

and $u_\beta(x)$ is the function with Fourier representation

$$\hat{u}_{\beta,k} = \begin{cases} 0, & k = 0 \\ 2\beta^k, & k > 0 \\ 2\bar{\beta}^{|k|}, & k < 0 \end{cases}, \quad \beta \in \Delta = \{z : |z| < 1\}. \quad (4)$$

The function $u_\beta(x)$ has a peak centered at $x = \arg(\bar{\beta})$ with amplitude growing to infinity as $|\beta|$ approaches 1. The N -hump traveling waves are a special case of these particle solutions:

$$u_{\alpha_0, N, \beta}(x, t) = \alpha_0 + \sum_{l=1}^N u_{\beta_l(t)}(x), \quad \beta_l(t) = \sqrt[N]{\beta} e^{-ict}, \quad c = \alpha_0 - N\alpha(\beta). \quad (5)$$

Each β_l is assigned a distinct N th root of β and $\alpha(\beta)$ is the mean of the one-hump stationary solution, namely

$$\alpha(\beta) = \frac{1 - 3|\beta|^2}{1 - |\beta|^2}, \quad |\beta|^2 = \frac{1 - \alpha(\beta)}{3 - \alpha(\beta)}. \quad (6)$$

The solution (5) moves to the right when $c > 0$. Indeed, it may also be written

$$u_{\alpha_0, N, \beta}(x, t) = u_{N, \beta}(x - ct) + c, \quad (7)$$

where $u_{N, \beta}$ is the N -hump stationary solution

$$u_{N, \beta}(x) = N\alpha(\beta) + \sum_{\{\gamma : \gamma^N = \beta\}} u_\gamma(x) = N\alpha(\beta) + Nu_\beta(Nx). \quad (8)$$

The Fourier representation of $u_{N, \beta}$ is

$$\hat{u}_{N, \beta, k} = \begin{cases} N\alpha(\beta), & k = 0, \\ 2N\beta^{k/N}, & k \in N\mathbb{Z}, k > 0, \\ 2N\bar{\beta}^{|k|/N}, & k \in N\mathbb{Z}, k < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Amick and Toland have shown [3] that all traveling waves of the Benjamin-Ono equation have the form (7); see also [31].

2.2 Linearization about Stationary Solutions

Let $u = u_{N, \beta}$ be an N -hump stationary solution. In [2], we solved the linearized equation

$$v_t = H v_{xx} - (uv)_x = iBAv, \quad A = H\partial_x - u, \quad B = \frac{1}{i}\partial_x \quad (10)$$

by solving the eigenvalue problem

$$BAz = \omega z \quad (11)$$

in closed form. Specifically, we showed that the eigenvalues $\omega_{N,n}$ are given by

$$\omega_{N,n} = \begin{cases} -\omega_{N,-n} & n < 0, \\ 0 & n = 0, \\ (n)(N-n) & 1 \leq n \leq N-1, \\ (n+1-N)(n+1+N(1-\alpha(\beta))) & n \geq N. \end{cases} \quad (12)$$

The zero eigenvalue $\omega_{N,0} = 0$ has geometric multiplicity two and algebraic multiplicity three. The eigenfunctions in the kernel of BA are

$$z_{N,0}^{(1,0)}(x) = -\frac{\partial}{\partial x} u_{N,\beta}(x), \quad z_{N,0}^{(2)}(x) = \frac{\partial}{\partial |\beta|} u_{N,\beta}(x), \quad (13)$$

which correspond to changing the phase or amplitude of β in the underlying stationary solution $u_{N,\beta}(x)$. There is also a Jordan chain [30] of length two associated with $z_{N,0}^{(1,0)}(x)$, namely

$$z_{N,0}^{(1,1)}(x) = 1, \quad \left(iBAz_{N,0}^{(1,1)} = z_{N,0}^{(1,0)} \right), \quad (14)$$

which corresponds to the fact that adding a constant to a stationary solution causes it to travel; see [2]. The eigenfunctions $z_{N,n}(x)$ corresponding to positive eigenvalues $\omega_{N,n}$ (with $n \geq 1$) have the Fourier representation

$$\begin{aligned} \hat{z}_{N,n,k} \Big|_{k=n+jN} &= \begin{cases} \left(1 + \frac{N(|j|-1)}{N-n} \right) \bar{\beta}^{|j|-1} & j < 0 \\ C \left(1 + \frac{Nj}{n} \right) \beta^{j+1} & j \geq 0 \end{cases}, \quad \left(C = \frac{1 \leq n \leq N-1}{(N-n)[n+(N-n)|\beta|^2]} \right), \\ \hat{z}_{N,n,k} \Big|_{k=n+1-N+jN} &= \begin{cases} 0 & j < 0 \\ \frac{-\bar{\beta}}{(1-|\beta|^2)^2} \left[1 - \left(1 - \frac{N}{n+1} \right) |\beta|^2 \right] & j = 0 \\ \left(1 + \frac{N(j-1)}{n+1} \right) \beta^{j-1} & j > 0 \end{cases}, \quad (n \geq N), \end{aligned} \quad (15)$$

with all other Fourier coefficients equal to zero. The eigenfunctions corresponding to negative eigenvalues $\omega_{N,n}$ (with $n \leq -1$) satisfy $z_{N,n}(x) = \overline{z_{N,-n}(x)}$, so the Fourier coefficients appear in reverse order, conjugated. For $1 \leq n \leq N-1$, any linear combination of $z_{N,n}(x)$ and $z_{N,N-n}(x)$ is also an eigenfunction; however, the choices here seem most natural as they simultaneously diagonalize the shift operator (discussed below) and yield directions along which non-trivial solutions exist beyond the linearization. Said differently, we have listed the first $N-1$ positive eigenvalues $\omega_{N,n}$ in an unusual order (rather than enumerating them monotonically and coalescing multiple eigenvalues) because this is the order that leads to the simplest description of the global paths of non-trivial solutions connecting these traveling waves.

2.3 Classification of bifurcations from traveling waves

Time-periodic solutions of the Benjamin-Ono equation with period T have initial conditions that satisfy $F(u_0, T) = 0$, where $F : H^1 \times \mathbb{R} \rightarrow H^1$ is given by

$$F(u_0, T) = u(\cdot, T) - u_0, \quad u_t = Hu_{xx} - uu_x, \quad u(\cdot, 0) = u_0. \quad (16)$$

We continue to assume that u is an N -hump stationary solution so that $DF = (D_1F, D_2F) : H^1 \times \mathbb{R} \rightarrow H^1$ satisfies

$$\begin{aligned} D_1F(u, T)v_0 &= \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} F(u + \varepsilon v_0, T) = v(\cdot, T) - v_0 = [e^{iBAT} - I]v_0, \\ D_2F(u, T)\tau &= \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} F(u, T + \varepsilon\tau) = 0. \end{aligned} \quad (17)$$

Note that $v_0 \in \ker D_1F(u, T)$ iff the solution $v(x, t)$ of the linearized problem is periodic with period T . As a result, a basis for the kernel $\mathcal{N} = \ker DF(u, T)$ consists of $(0; 1)$ together with all pairs $(v_0; 0)$ of the form

$$v_0(x) = \operatorname{Re}\{z_{N,n}(x)\} \quad \text{or} \quad v_0(x) = \operatorname{Im}\{z_{N,n}(x)\}, \quad (18)$$

where n ranges over all integers such that

$$\omega_{N,n}T \in 2\pi\mathbb{Z} \quad (19)$$

with N and β (in the formula (12) for $\omega_{N,n}$) held fixed. The corresponding periodic solutions of the linearized problem are

$$v(x, t) = \operatorname{Re}\{z_{N,n}(x)e^{i\omega_{N,n}t}\} \quad \text{or} \quad v(x, t) = \operatorname{Im}\{z_{N,n}(x)e^{i\omega_{N,n}t}\}. \quad (20)$$

Negative values of n have already been accounted for in (18) and (20) using $z_{N,-n}(x) = \overline{z_{N,n}(x)}$, and the $n = 0$ case always yields two vectors in the kernel, namely those in (13). These directions do not cause bifurcations as they lead to other stationary solutions.

Next we wish to linearize F about an arbitrary traveling wave. Suppose $u = u_{N,\beta}(x)$ is an N -hump stationary solution and $U(x, t) = u(x - ct) + c$ is a traveling wave. Then the solutions v and V of the linearizations about u and U , respectively, satisfy $V(x, t) = v(x - ct, t)$. Note also that

$$F(U_0, T) = 0 \quad \text{iff} \quad cT = \frac{2\pi\nu}{N} \quad \text{for some } \nu \in \mathbb{Z}, \quad (21)$$

where $U_0(x) = U(x, 0) = u(x) + c$. Assuming $\theta = cT$ belongs to $\frac{2\pi}{N}\mathbb{Z}$, we have

$$\begin{aligned} [D_1F(U_0, T)v_0](x) &= v(x - cT, T) - v_0(x) = [(S_\theta e^{iBAT} - I)v_0](x), \\ [D_2F(U_0, T)\tau](x) &= U_t(x, T)\tau = -cu_x(x - cT)\tau = -cu_x(x)\tau, \end{aligned} \quad (22)$$

where v solves (10) and the shift operator S_θ is defined via

$$S_\theta z(x) = z(x - \theta), \quad \hat{S}_{\theta,kl} = e^{-ik\theta} \delta_{kl}. \quad (23)$$

One element of $\mathcal{N} = \ker DF(U_0, T)$ arises from (14), which gives

$$e^{iBAT} 1 = 1 - tu_x \quad \Rightarrow \quad D_1 F(U_0, T)(-c/T) + D_2 F(U_0, T)1 = 0,$$

and implies $(-c/T; 1) \in \mathcal{N}$. This just means that we can change the period T by a small amount τ by adding the constant $-(c/T)\tau$ to U_0 ; (this also follows from the condition (21) that $cT = \theta = \text{const}$). If we wish to change the period without changing the mean, we need to simultaneously adjust $|\beta|$ in the underlying stationary solution $u = u_{N,\beta}$. The other elements of \mathcal{N} are of the form $(v_0, 0)$ with

$$v_0(x) = \text{Re}\{z_{N,n}(x)\} \quad \text{or} \quad v_0(x) = \text{Im}\{z_{N,n}(x)\}. \quad (24)$$

The admissible values of n here are found using (22) together with

$$S_\theta e^{iBAT} z_{N,n} = e^{i(\omega_{N,n}T - \theta k_{N,n})} z_{N,n}, \quad \theta = \frac{2\pi\nu}{N}, \quad (25)$$

where $k_{N,n}$ is the stride offset of the non-zero Fourier coefficients of $z_{N,n}$, i.e.

$$\hat{z}_{N,n,k} \neq 0 \quad \Rightarrow \quad k - k_{N,n} \in N\mathbb{Z}. \quad (26)$$

Thus, instead of (19), n ranges over all integers such that

$$\omega_{N,n}T \in 2\pi \left(\frac{\nu k_{N,n}}{N} + \mathbb{Z} \right), \quad k_{N,n} = \begin{cases} -k_{N,-n}, & n < 0, \\ 0 & n = 0, \\ n & 1 \leq n \leq N-1, \\ \text{mod}(n+1, N) & n \geq N. \end{cases} \quad (27)$$

As before, negative values of n need not be considered once we take real and imaginary parts in (24), and the $n = 0$ case always gives the two vectors $(z_{N,0}^{(1,0)}, 0), (z_{N,0}^{(2)}, 0) \in \mathcal{N}$, which lead to other traveling waves rather than bifurcations to non-trivial solutions.

Our numerical experiments have led us to the following conjecture, which has now been proved; see [31]:

CONJECTURE 1 *For every $\beta \in \Delta$ and $(N, \nu, n, m) \in \mathbb{Z}^4$ satisfying*

$$N \geq 1, \quad \nu \in \mathbb{Z}, \quad n \geq 1, \quad m \geq 1, \quad m \in \nu k_{N,n} + N\mathbb{Z}, \quad (28)$$

there is a four parameter sheet of non-trivial time-periodic solutions bifurcating from the N -hump traveling wave with speed index ν , ($cT = 2\pi\nu/N$), bifurcation index n , (discussed below), oscillation index m , ($\omega_{N,n}T = 2\pi m/N$), and phase and amplitude governed by β .

The main content of this conjecture is that we do not have to consider linear combinations of the $z_{N,n}$ with different values of n to find periodic solutions of the non-linear problem — this basis is already “diagonal” with respect to these bifurcations. This is true even when the nullspace \mathcal{N} of $DF(U_0, T)$ is infinite dimensional, i.e. when the parameter $\alpha = (1 - 3|\beta|^2)/(1 - |\beta|^2)$ in the formula (12) for $\omega_{N,n}$ is rational so that there are infinitely many values of n satisfying (27). The decision to number the first $N - 1$ eigenvalues $\omega_{N,n}$ non-monotonically in (12) and to simultaneously diagonalize the shift operator S_θ when choosing eigenvectors $z_{N,n}$ in (15) was essential to make this work.

A canonical way to generate one of these bifurcations is to take β real and perturb the initial condition in the direction $v_0(x) = \text{Re}\{z_{N,n}(x)\}$. This leads to non-trivial solutions with even symmetry at $t = 0$. Perturbation in the $\text{Im}\{z_{N,n}(x)\}$ direction yields the same set of non-trivial solutions, but with a spatial and temporal phase shift:

$$\text{Im}\{z_{N,n}(x - ct)e^{i\omega t}\} = \text{Re}\left\{z_{N,n}\left(\left(x - \frac{c\pi}{2\omega}\right) - c\left(t - \frac{\pi}{2\omega}\right)\right)e^{i\omega\left(t - \frac{\pi}{2\omega}\right)}\right\}, \quad (\omega = \omega_{N,n}).$$

The manifold of non-trivial solutions is four dimensional with two essential parameters (e.g. the mean α_0 and a parameter governing the distance from the traveling wave) and two inessential parameters (the spatial and temporal phase). In our numerical studies, we use the real part of a Fourier coefficient c_k of the initial condition (with k such that $\hat{z}_{N,n,k} \neq 0$) for the second essential bifurcation parameter. When we discuss exact solutions in Section 4, we propose an ansatz for the form of these non-trivial solutions and use a coefficient in the ansatz for this parameter.

We remark that this enumeration of bifurcations accounts for all time-periodic solutions of the linearization about traveling waves; therefore, the heuristic that each bifurcation of the non-linear problem gives rise to a linearly independent vector in the kernel \mathcal{N} of the linearized problem suggests that we have found all bifurcations from traveling waves. Interestingly, this turns out not to be the case; the interior bifurcations we discuss in Section 5 can occur at the endpoints of the path, allowing for degenerate bifurcations directly from traveling waves to higher levels in the infinite hierarchy of time-periodic solutions. Only the transition from the first level of the hierarchy to the second is “visible” to a linearized analysis about traveling waves; the other transitions are be discussed in [31].

In computing the nullspace \mathcal{N} above, we considered N, ν, β, T (and hence α_0) to be given and searched for compatible indices n and m . The decay parameter $|\beta|$, the mean α_0 , and the period T cannot be specified independently; any two of them determines the third. We now derive formulas for the period and mean in terms of (N, ν, n, m) and β . To simplify the formulas, we work with $\alpha = (1 - 3|\beta|^2)/(1 - |\beta|^2)$ instead of β . Note that as

we increase $|\beta|$ from 0 to 1, α decreases from 1 to $-\infty$. For the period, we have

$$T = \frac{2\pi m}{N\omega_{N,n}} = \begin{cases} \frac{2\pi m}{Nn(N-n)} & n < N, \\ \frac{2\pi m}{N(n+1-N)(n+1+N(1-\alpha))} & n \geq N, \end{cases} \quad (29)$$

so the period is independent of β when $n < N$, and otherwise decreases to zero as $|\beta|$ varies from 0 to 1. For the mean, α_0 , we note that

$$cT = \frac{2\pi\nu}{N}, \quad c = \alpha_0 - N\alpha \quad \Rightarrow \quad \alpha_0 = N\alpha + \frac{2\pi\nu}{NT}. \quad (30)$$

Hence, using $\frac{2\pi}{NT} = \frac{\omega_{N,n}}{m}$, we obtain

$$\alpha_0 = \begin{cases} N + \frac{n(N-n)}{m}\nu - (1-\alpha)N, & n < N, \\ N + \frac{(n+1-N)(n+1)}{m}\nu - \left(1 - \frac{n+1-N}{m}\nu\right)N(1-\alpha), & n \geq N. \end{cases} \quad (31)$$

Thus, as $|\beta|$ varies from 0 to 1, the mean α_0 decreases to $-\infty$ if $n < N$, and otherwise either decreases to $-\infty$, increases to $+\infty$, or is independent of β , depending on the sign of $[m - (n+1-N)\nu]$.

In practice, we often wish to start with N , ν , n , m and α_0 and determine T and $|\beta|$ from these. However, not all values of α_0 are compatible with a given set of indices. The bifurcation rules are summarized in Figure 1. Solving (31) for α yields

$$\alpha = \begin{cases} 1 - \frac{(N - \alpha_0)m + n(N - n)\nu}{Nm}, & n < N, \\ 1 - \frac{(N - \alpha_0)m + (n + 1 - N)(n + 1)\nu}{[m - (n + 1 - N)\nu]N}, & n \geq N. \end{cases} \quad (32)$$

The corresponding period is given by

$$T = \begin{cases} \frac{2\pi m}{Nn(N-n)}, & n < N, \\ \frac{2\pi \left(\frac{m}{n+1-N} - \nu\right)}{N(n+1+N-\alpha_0)}, & n \geq N. \end{cases} \quad (33)$$

In the indeterminate cases $\{n \geq N, m = (n+1-N)\nu, \alpha_0 = n+1+N\}$, any $\alpha \leq 1$ is allowed and formula (29) should be used to determine T .

3 Numerical Experiments

In this section we present a collection of numerical experiments in which we start with a given bifurcation (N, ν, n, m, β) and use a variant of the method we described in [2] for finding

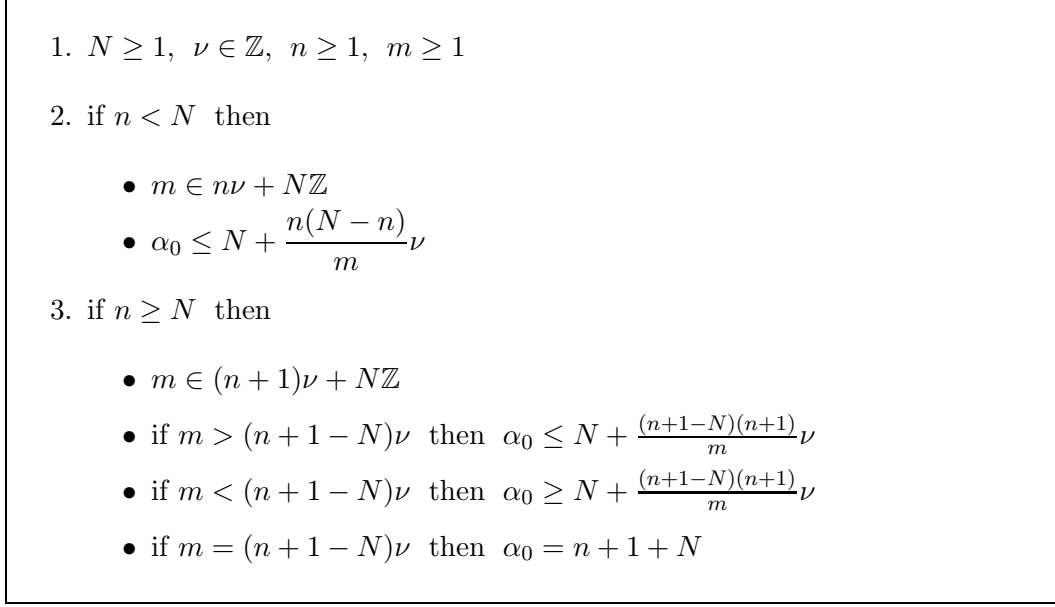


Figure 1: Bifurcation rules governing which values of α_0 are compatible with (N, ν, n, m) .

periodic solutions of non-linear PDE to continue these solutions until another traveling wave is found, or until the solution blows up as the bifurcation parameter approaches a critical value. We determine the bifurcation indices (N', ν', n', m') at the other end of the path of non-trivial solutions by fitting the data to the formulas of the previous section. By trial and error, we are then able to guess a formula relating (N', ν', n', m') to (N, ν, n, m) that we use in Section 4 to construct exact solutions.

3.1 Numerical Method

To find non-trivial time-periodic solutions of the Benjamin-Ono equation, we use a symmetric variant of the algorithm described in [2]. Here we take advantage of the fact that if $u(x, t)$ is a solution of (1), then so is $U(x, t) = u(-x, -t)$. As a result, if u is a solution such that $u(x, T/2) = U(x, -T/2)$, then $u(x, T) = U(x, 0)$, i.e. u is time-periodic if the initial condition has even symmetry. Thus, we seek initial conditions u_0 with even symmetry and a period T to minimize the functional

$$G_{\text{tot}}(u_0, T) = G(u_0, T) + \varphi(u_0, T), \quad (34)$$

where

$$G(u_0, T) = \frac{1}{2} \int_0^{2\pi} [u(x, T/2) - u(2\pi - x, T/2)]^2 dx \quad (35)$$

and $\varphi(u_0, T)$ is a non-negative penalty function to impose the mean and set the bifurcation parameter. Although the original method works well, we used the symmetric variant for the simulations in this paper because evolving to $T/2$ requires half the time-steps and yields more accurate answers as there is less time for numerical roundoff error to corrupt the calculation. Moreover, the number of degrees of freedom in the search space of initial conditions u_0 is also cut in half and the condition number of the problem improves when we eliminate the phase degrees of freedom via symmetry rather than including them in the penalty function φ . In [2], we gave a heuristic argument based on Liapunov-Schmidt theory that suggests that all bifurcations from traveling waves should possess even symmetry at $t = 0$ (possibly after a spatial and temporal phase shift).

To compute the gradient of G with respect to variation of the initial conditions, we use

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} G(u_0 + \varepsilon v_0, T) = \int_0^{2\pi} \frac{\delta G}{\delta u_0}(x) v_0(x) dx, \quad (36)$$

where the variational derivative

$$\frac{\delta G}{\delta u_0}(x) = 2w(x, T/2), \quad w_0(x) = u(x, T/2) - u(2\pi - x, T/2) \quad (37)$$

is found by solving the following adjoint equation from $s = 0$ to $s = T/2$:

$$w_s(x, s) = -H w_{xx}(x, s) + u(x, \frac{T}{2} - s) w_x(x, s), \quad w(\cdot, 0) = w_0. \quad (38)$$

Since v_0 is assumed symmetric in this formulation, (37) is equivalent to

$$\frac{\delta G}{\delta u_0}(x) = w(x, T/2) + w(2\pi - x, T/2). \quad (39)$$

The Benjamin-Ono and adjoint equations are solved using a pseudo-spectral collocation method employing a fourth order semi-implicit additive Runge-Kutta method [11, 18, 32] to advance the solution in time. The BFGS method [7, 24] is then used to minimize G_{tot} (varying the period and the Fourier coefficients of the initial conditions). We use the penalty function

$$\varphi(u_0, T) = \frac{1}{2} \left([a_0(0) - \alpha_0]^2 + [a_K(0) - \rho]^2 \right) \quad (40)$$

to specify the mean α_0 and the real part ρ of the K th Fourier coefficient of the initial condition

$$u_0(x) = \sum_{k=-M/2+1}^{M/2} c_k(0) e^{ikx}, \quad c_k(t) = a_k(t) + ib_k(t). \quad (41)$$

The parameters α_0 and ρ serve as the bifurcation parameters while the phases are determined by requiring that the solution have even symmetry at $t = 0$. We generally choose

K to be the first $k \geq 1$ such that $\hat{z}_{N,n,k} \neq 0$. See [2] for a discussion of our motivation for using $\rho = a_K(0)$ as a bifurcation parameter.

We now describe our continuation method. The solutions in the bifurcation diagram of Figure 2 below were found by holding the mean $\alpha_0 = 0.544375$ constant and varying the value ρ of the second Fourier coefficient $a_2(0)$ using the penalty function φ . The initial guess was obtained by perturbing the initial condition $U_0(x) = u_{N,\beta}(x) + c$ of the appropriate stationary or traveling wave in the direction $z_{N,n}(x)$ predicted by linear theory. We use the period T given in (29) above as a starting guess. For each solution found, we record its period T and plot it in the diagram. As we vary the parameter ρ , we use linear extrapolation for the initial guess of the next minimization step. If the initial value of G_{tot} is too large, we discard the step and try again with a smaller change in ρ .

The running time of our algorithm (on a 2.4 GHz desktop machine) varies from a few hours to compute one of the paths labeled a – l in (42)–(45) below, to a few days to compute a path in which the solution blows up, such as the one shown in Figure 6 below. We always refine the mesh and timestep enough so that the solutions are essentially exact (with $G_{\text{tot}} \leq 10^{-26}$ in the easy cases and 10^{-20} in the hard cases).

3.2 Global paths of non-trivial solutions

We now investigate the global behavior of non-trivial solutions that bifurcate from arbitrary stationary or traveling waves. We find that these non-trivial solutions act as rungs in a ladder, connecting stationary and traveling solutions with different speeds and wavelengths by creating or annihilating oscillatory humps that grow or shrink in amplitude until they become part of the stationary or traveling wave on the other side of the rung. In some cases, rather than re-connecting with another traveling wave, the solution blows up (i.e. the L^2 -norm of the initial condition grows without bound) as the bifurcation parameter ρ approaches a critical value. However, even in these cases a re-connection with another traveling wave does occur if, in addition to ρ , we vary the mean, α_0 , in an appropriate way.

Recall from Section 2.3 that we can enumerate all such bifurcations by specifying a complex parameter β in the unit disk Δ along with four integers (N, ν, n, m) satisfying (28), and in most cases we can solve for $|\beta|$ in terms of the mean, α_0 , using (32). In [2], we presented a detailed study of the solutions on the path connecting a one-hump stationary solution to a two-hump traveling wave moving left. We denote this path by

$$a : (1, 0, 1, 1) \longleftrightarrow (2, -1, 1, 1), \quad (42)$$

where the label a refers to the bifurcation diagram in Figure 2. We have also computed the next several bifurcations ($n = 2, 3, 4$) from the one-hump stationary solution and found that

they connect up with a traveling wave with $N' = n + 1$ humps moving left with speed index $\nu' = -1$, where we denote the bifurcation on the other side of the path by (N', ν', n', m') . By comparing the Fourier coefficients of the last few non-trivial solutions on this path to those of the linearization about the N' -hump traveling wave, we determined that the bifurcation and oscillation indices satisfy $n' = n$ and $m' = 1$, respectively. These results are summarized as

$$\begin{aligned} b : \quad (1, 0, 2, 1) & \longleftrightarrow (3, -1, 2, 1), \\ c : \quad (1, 0, 3, 1) & \longleftrightarrow (4, -1, 3, 1), \\ d : \quad (1, 0, 4, 1) & \longleftrightarrow (5, -1, 4, 1). \end{aligned} \tag{43}$$

A solution on path d is shown in Figure 3. It was not obvious at first that the correct way to number the eigenvalues $\omega_{N', n'}$ was to split the double eigenvalues with $n' < N'$ apart as we did in (12) by simultaneously diagonalizing the shift operator and ordering the $\omega_{N', n'}$ via the stride offset of the corresponding eigenvectors (rather than monotonically). But using this ordering, the double-eigenvalues bifurcate as if they were simple eigenvalues, i.e. the non-trivial solutions connect up with the N' -hump traveling wave along the $z_{N', n'}$ direction (without involving $z_{N', N' - n'}$).

The labels a , b , c , d in (42) and (43) correspond to the paths labeled $7d$, $8d$, $5c$, a , etc. in the bifurcation diagram of Figure 2. When an integer p precedes a label, it means that the period T that is plotted is p times larger than the fundamental period of the solution represented. In our labeling scheme, we just need to multiply ν , m , ν' , m' by p to obtain the new path, e.g.

$$7d : \quad (1, 0, 4, 7) \longleftrightarrow (5, -7, 4, 7). \tag{44}$$

In this diagram, we plot $a_2(0)$ vs. T with the spatial and temporal phases chosen so the solution is even at $t = 0$. It is interesting that the paths labeled a and b meet the one-hump stationary solutions in a pitchfork while the other paths c and d meet at an oblique angle from one side only. This is because the second Fourier mode of the eigenvector $z_{1, n}(x)$ in the linearization about the stationary solution is zero in the latter two cases, so the change in $a_2(0)$ from that of the stationary solution (namely 0.371087) is a higher order effect. When we go beyond the linearization as we have here, we find that $c_2(t) = a_2(t) + ib_2(t)$ has a nearly circular (epitrochoidal) orbit in case a , a circular orbit in case b , and remains constant in time in cases c and d ; see Section 4. If one branch of the pitchfork corresponds to $a_2(0)$, the other is $a_2(T/2)$ — the function $u(\cdot, T/2)$ also has even symmetry. But in cases c and d , $a_2(0)$ is equal to $a_2(T/2)$ even though the functions $u(\cdot, 0)$ and $u(\cdot, T/2)$ are different. These cases also become pitchforks when a different Fourier coefficient $a_K(0)$ is used as the bifurcation parameter.

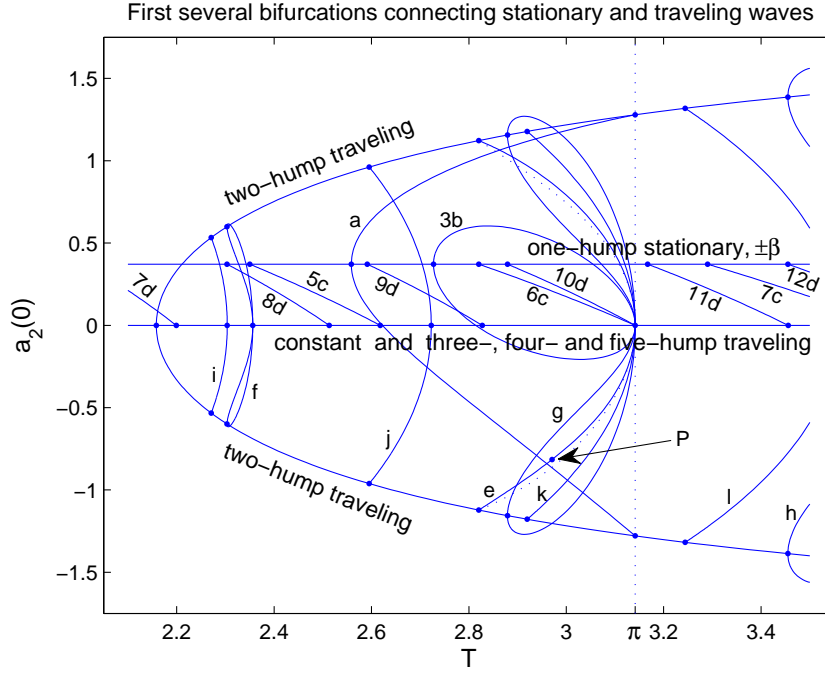


Figure 2: Paths of non-trivial solutions listed in equations (42)–(45). The second Fourier mode of the eigenvector $z_{N,n}(x)$ in the linearization is non-zero for the pitchfork bifurcations and is zero for the one-sided, oblique-angle bifurcations.

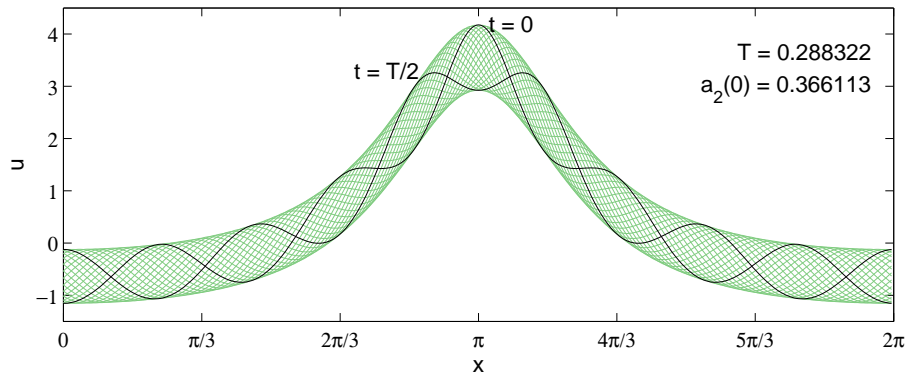


Figure 3: Periodic solution on path d connecting the one-hump stationary solution to the five-hump left-traveling wave ($\alpha_0 = 0.544375$). The second Fourier mode of $z_{1,4}(x)$ is zero, which explains why $a_2(0) = 0.366113$ for this solution is only 1.35% of the way between the stationary solution $a_2(0) = 0.371087$ and the five-hump traveling wave $a_2(0) = 0$.

Next we compute the first several bifurcations from the two-hump traveling waves with mean $\alpha_0 = 0.544375$ and speed index $\nu = -1$. We set $N = 2$, $\nu = -1$, $n \in \{1, 2, 3, 4\}$ and choose the first several legal m values, i.e. values of m that satisfy the bifurcation rules of Figure 1. For example, the curves labeled i , j , k and l in Figure 2 correspond to the bifurcations $(2, -1, 4, m)$ with $m = 11, 13, 15, 17$; smaller values (and even values) of m are not allowed. In addition to the path a in (42) above, we obtain the paths

$$\begin{aligned}
e: & \quad (2, -1, 2, 3) \longleftrightarrow (3, -3, 1, 3), & i: & \quad (2, -1, 4, 11) \longleftrightarrow (5, -8, 3, 11), \\
f: & \quad (2, -1, 3, 6) \longleftrightarrow (4, -5, 2, 6), & j: & \quad (2, -1, 4, 13) \longleftrightarrow (5, -9, 3, 13), \\
g: & \quad (2, -1, 3, 8) \longleftrightarrow (4, -6, 2, 8), & k: & \quad (2, -1, 4, 15) \longleftrightarrow (5, -10, 3, 15), \\
h: & \quad (2, -1, 3, 10) \longleftrightarrow (4, -7, 2, 10), & l: & \quad (2, -1, 4, 17) \longleftrightarrow (5, -11, 3, 17).
\end{aligned} \tag{45}$$

The paths f , g and h meet the curve representing the two-hump traveling waves in a pitchfork bifurcation while the others meet obliquely from one side. This, again, is an anomaly of having chosen the second Fourier mode for the bifurcation parameter. The dotted line near the path e is the curve obtained when e is reflected across the x -axis. Solutions on this dotted line correspond to solutions on path e shifted by $\pi/2$ in space, which changes the sign of $\rho = a_2(0)$ but also breaks the even symmetry of the solution at $t = 0$. The paths labeled i , j , k and l are exactly symmetric because $c_2(t)$ has a circular orbit centered at zero in these cases. It is interesting that so many of the paths in this bifurcation diagram terminate when $T = \pi$ (or a simple rational multiple of π). This is due to the fact that T in (29) is independent of α when $n < N$.

The solutions $u(x, t)$ corresponding to points along the paths b , c and d are qualitatively similar to each other; see Figure 3. These solutions look like N' -hump waves traveling over a stationary one-hump carrier signal. At one end of the path the high frequency wave may be viewed as a perturbation of the one-hump stationary solution, while at the other end of the path it is more appropriate to regard the stationary solution as the perturbation, causing the traveling wave to bulge upward as it passes near $x = \pi$ and downward near $x = 0$ and $x = 2\pi$. In all these cases, the solution repeats itself when one of the high frequency waves has moved left one slot to assume the shape of its left neighbor at $t = 0$.

By contrast, the solutions that bifurcate from the two-hump traveling waves, i.e. those on the paths listed in (45), have the property that when a wave has moved left one slot to the location that its neighbor occupied at $t = 0$, it has acquired a different shape and must keep progressing a number of slots before it finally lines up with one of the initial waves. This is illustrated in Figure 4 for the solution labeled P in Figure 2 on the path

$$e: \quad (2, -1, 2, 3) \longleftrightarrow (3, -3, 1, 3). \tag{46}$$

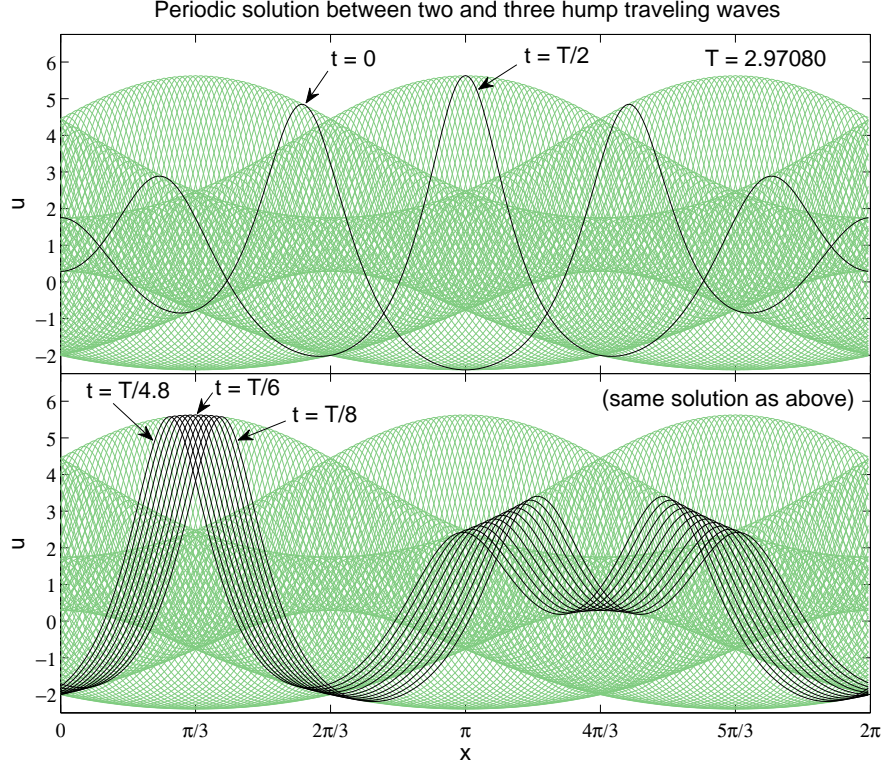


Figure 4: Time-periodic solution (labeled P in figure 2) on path e connecting two- and three-hump traveling waves. The amplitude of each hump oscillates as it travels left.

This solution is qualitatively similar to the linearized solution $(3, -3, 1, 3)$; there are $N' = 3$ humps oscillating with the same amplitude but with different phases as they travel left. They do not line up with the initial condition again until they have traveled three slots ($\nu' = -3$) and progressed through one cycle ($m'/N' = 3/3$), which leads to a braided effect when the time history of the solution is plotted on one graph. All the solutions on path e are *irreducible* in the sense that there is no smaller time T in which they are periodic (unlike the cases labeled $3b$, $5c$, $7d$ etc. in Figure 2, which are reducible to b , c and d , respectively). Note that although $\nu' = -3$ and $m' = 3$ are both divisible by 3, we cannot reduce $(3, -3, 1, 3)$ to $(3, -1, 1, 1)$ as the latter indices violate the bifurcation rules of Figure 1. We also mention that at the beginning of the path, near $(2, -1, 2, 3)$, the braiding effect is not present; instead, the solution can be described as two humps bouncing out of phase as they travel left. In one period, they each travel left one slot ($\nu = -1$) and bounce 1.5 times ($m/N = 3/2$) to assume the shape of the other hump at $t = 0$. The transition from this behavior to the braided behavior occurs at the point on path e that a third hump

becomes recognizable in the wave profile. The solutions on the paths f, g, h, i, j, k and l are similar to those on path e , but the braiding patterns are more complicated near the right end-points of these paths.

All the traveling waves we have described until now move left. To see what happens to a right-moving wave, we computed the first bifurcation from the simplest such case and obtained the path

$$(1, 1, 1, 2) \longleftrightarrow (2, 0, 1, 2). \quad (47)$$

Thus, the one-hump right-traveling wave is connected to the two-hump stationary solution. Solutions near the left end of this path consist of a large-amplitude, right-moving soliton traveling over a small-amplitude, left-moving soliton. As we progress along the path, the amplitude of the left-moving soliton increases until the solitons cease to fully merge at $t = T/4$ and $t = 3T/4$. Instead, a dimple forms in the wave profile at these times and the solitons begin to bounce off each other, trading amplitude so the right-moving wave is larger than the left-moving wave. This type of behavior has also been observed by Leveque [19] for the KdV equation for solitons of nearly equal amplitude. Both types of behavior (merging and bouncing off one another) are illustrated in Figure 5. As we proceed further along this path, the solitons settle into a synchronized dancing motion without changing their shape or deviating far from their initial positions. Eventually the “dancing amplitude” becomes small and the non-trivial solution turns into a stationary two-hump solution.

In order to guess a general formula for the relationship between two traveling waves that are connected by a path of non-trivial solutions, we generated two additional paths, namely

$$\begin{aligned} (2, 0, 2, 2) &\longleftrightarrow (3, -1, 1, 2), \\ (3, 0, 3, 3) &\longleftrightarrow (4, -1, 1, 3). \end{aligned} \quad (48)$$

After studying all the paths listed in (42)–(48), we propose the following conjecture, which has now been proved in [31]:

CONJECTURE 2 *The four-parameter sheet of non-trivial solutions with bifurcation parameters (N, ν, n, m) coincides with the sheet with parameters (N', ν', n', m') if and only if*

$$\text{if } n < N : \quad N' = N - n, \quad \nu' = \frac{(N - n)\nu + m}{N}, \quad n' = N - 1, \quad m' = m, \quad (49)$$

$$\text{if } n \geq N : \quad N' = n + 1, \quad \nu' = \frac{(n + 1)\nu - m}{N}, \quad n' = n + 1 - N, \quad m' = m. \quad (50)$$

By symmetry, we may interchange the primed and unprimed indices in either formula; thus, $N' > N \Leftrightarrow n < N \Leftrightarrow n' \geq N'$. Equations (49) and (50) are consistent with the

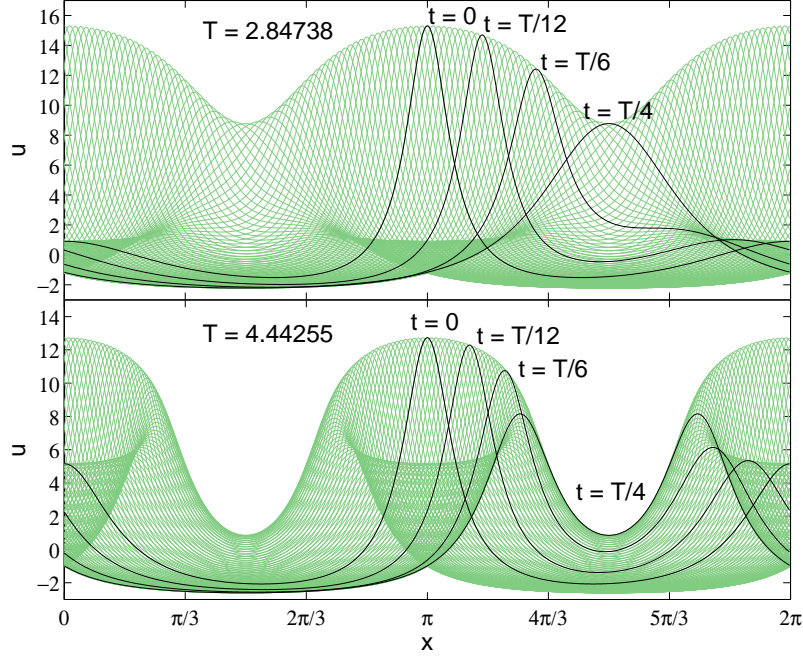


Figure 5: Periodic solutions between the one-hump right-traveling wave and the two-hump stationary solution ($\alpha_0 = 0.544375$). *Top*: a large, right-traveling soliton temporarily merges with a small, left traveling soliton at $t = \frac{T}{4}$ and $t = \frac{3}{4}T$. *Bottom*: two solitons traveling in opposite directions bounce off each other at $\frac{T}{4}$ and $\frac{3}{4}T$ and change direction.

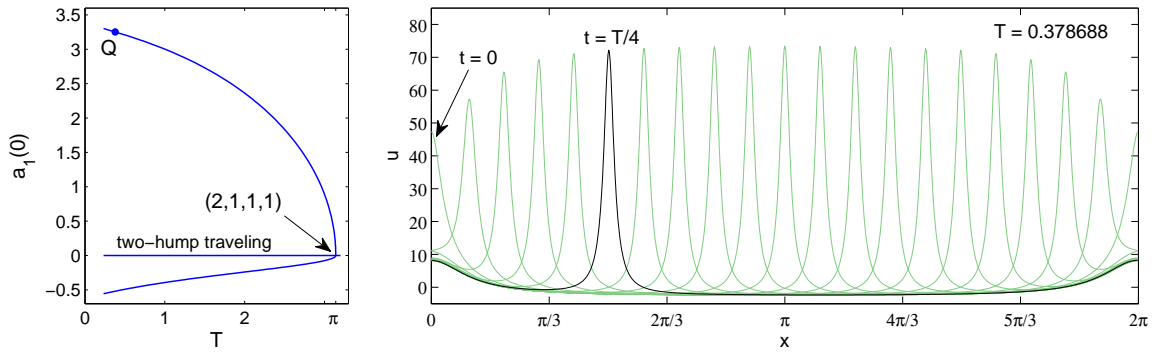


Figure 6: *Left*: path of non-trivial solutions with mean $\alpha_0 = 1.2$ that bifurcates with indices $(2, 1, 1, 1)$ from the two-hump traveling wave. These solutions do not re-connect with another traveling wave; they blow up as $T \rightarrow 0$. The solution labeled Q is shown at right, where a large, right-moving soliton travels rapidly over a small, stationary hump.

bifurcation rules in that

$$n < N, \quad m \in n\nu + N\mathbb{Z} \quad \Rightarrow \quad \nu' \in \mathbb{Z}, \quad m' \in (n' + 1)\nu' + N'\mathbb{Z}, \quad (51)$$

$$n \geq N, \quad m \in (n + 1)\nu + N\mathbb{Z} \quad \Rightarrow \quad \nu' \in \mathbb{Z}, \quad m' \in n'\nu' + N'\mathbb{Z}. \quad (52)$$

However, if the mean is held constant, they do not necessarily respect the requirements on α_0 listed in Figure 1. For example, if $\alpha_0 \leq 3$, then $(2, 1, 1, 1)$ is a valid bifurcation, but the re-connection $(1, 1, 1, 1)$ predicted by (49) is legal only if $\alpha_0 = 3$. Interestingly, when we use our numerical method to follow the path of non-trivial solutions that bifurcates from $(2, 1, 1, 1)$ with the mean $\alpha_0 = 1.2$ held constant, it does not connect up with another traveling wave. Instead, as we vary the bifurcation parameter, the two humps grow in amplitude and merge together (at $t = 0$) until they become a single soliton traveling very rapidly on top of a small amplitude stationary hump; see Figure 6. As the bifurcation parameter $\rho = a_1(0)$ approaches a critical value, the period T approaches zero and the solution blows up in $L^2(0, 2\pi)$ with the Fourier coefficients of any time-slice decaying more and more slowly.

As another example, the bifurcation $(3, 1, 1, 1)$ is valid when $\alpha_0 \leq 5$ but the reconnection $(2, 1, 2, 1)$ is only valid if $\alpha_0 = 5$. If we hold $\alpha_0 < 5$ constant, the solution blows up as we vary $\rho = a_2(0)$ from 0 to a critical value. However, if we simultaneously vary the mean α_0 so that it approaches 5, we do indeed reach a traveling wave with bifurcation indices $(2, 1, 2, 1)$. To check this numerically, we started at $(3, 1, 1, 1)$ with $\alpha_0 = 4.8$ (which has $\alpha = \frac{14}{15}$, $|\beta| = 1/\sqrt{31}$) and computed 40 solutions varying ρ from 0 to 0.1 and setting $\alpha_0 = 4.8 + 2\rho$. The bifurcation at the other end turned out to be $(2, 1, 2, 1)$ with $\alpha_0 = 5$, $\beta = \frac{1}{4}\rho = 0.025$, $\alpha = (1 - 3\beta^2)/(1 - \beta^2)$, $T = \pi/(5 - 2\alpha)$, as expected. The solutions on this path have the interesting property that the envelope of the solution pinches off into a football shape at one point in the transition from the three-hump traveling wave to the two-hump traveling wave. Using a bracketing technique, we were able to find a solution such that the value of $u(0, t)$ remained constant in time to 8 digits of accuracy. The result is shown in Figure 7.

In summary, it appears that the family of bifurcations with indices (N, ν, n, m) is always connected to the family with indices (N', ν', n', m') given by (49) and (50) by a sheet of non-trivial solutions, but we often have to vary both the mean and a Fourier coefficient of the initial condition to achieve a re-connection. Thus, the manifold of non-trivial solutions is genuinely two-dimensional (and technically four dimensional if the phase shifts are included), i.e. some of its important properties cannot be seen if we hold the mean α_0 constant.

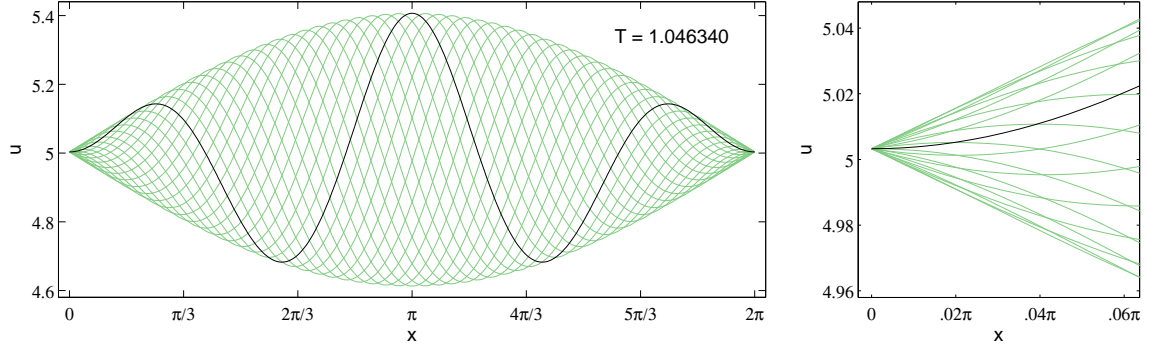


Figure 7: *Left:* One of the solutions on the path from $\{(3, 1, 1, 1), \beta = -\sqrt{\frac{1}{31}}\}$ to $\{(2, 1, 2, 1), \beta = \frac{1}{40}\}$ consists of a traveling wave inside a football-shaped envelope. The exact solution appears to be of the form $u(x, t) = A + B \left(\sin \frac{x}{2}\right) \sin \left(\frac{5}{2}x - \frac{2\pi}{T}t\right)$.

4 Exact Solutions

In this section we use data fitting techniques to determine the analytic form of the numerical solutions of Section 3. We also derive algebraic relationships between the unknown coefficients in the ansatz in order to explain why some paths of solutions reconnect and others lead to blow-up.

4.1 Fourier Coefficients and Lattice Sums

One striking feature of the time-periodic solutions we have found numerically is that the trajectories of the Fourier modes $c_k(t)$ are often circular or nearly circular. Other Fourier modes have more complicated trajectories resembling cardioids, flowers and many other familiar “spirograph” patterns; see Figure 8. This led us to experiment with data fitting to try to guess the analytic form of these solutions. The first thing we noticed was that the trajectories of the spatial Fourier coefficients are band-limited in time, with the width of the band growing linearly with the wave number:

$$u(x, t) = \sum_{k=-\infty}^{\infty} c_k(t) e^{ikx}, \quad c_k(t) = \sum_{j=-\infty}^{\infty} c_{kj} e^{-ij \frac{2\pi}{T} t}, \quad c_{kj} = 0 \text{ if } |j| > r|k|. \quad (53)$$

Here r is a fixed positive integer (depending on which path of non-trivial solutions u belongs to) and the c_{kj} are real numbers when a suitable choice of spatial and temporal phase is made. Since u is real, these coefficients satisfy $c_{-k, -j} = c_{kj}$.

Each path of non-trivial time-periodic solutions has a lattice pattern of non-zero Fourier coefficients c_{kj} associated with it. In Figure 9, we show the lattice of integers (k, j) such

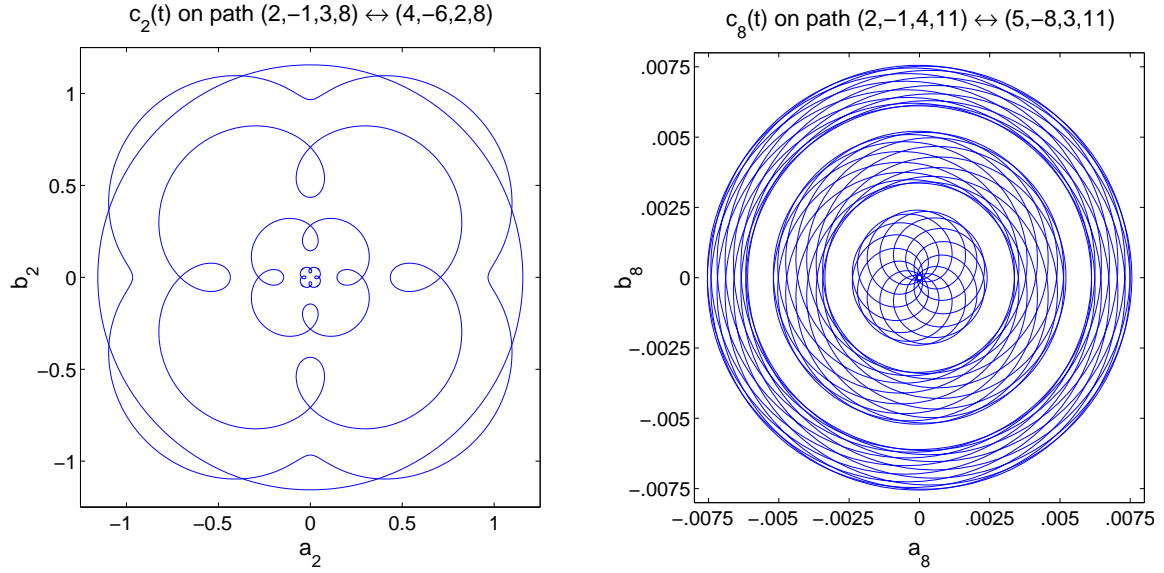


Figure 8: *Left:* Trajectories $c_2(t)$ for five solutions on path g in (45). The evolution of $c_2(t)$ on paths f and h in (45) are similar, but with three- and five-fold symmetry rather than four. *Right:* Trajectories $c_8(t)$ for three solutions on path i in (45).

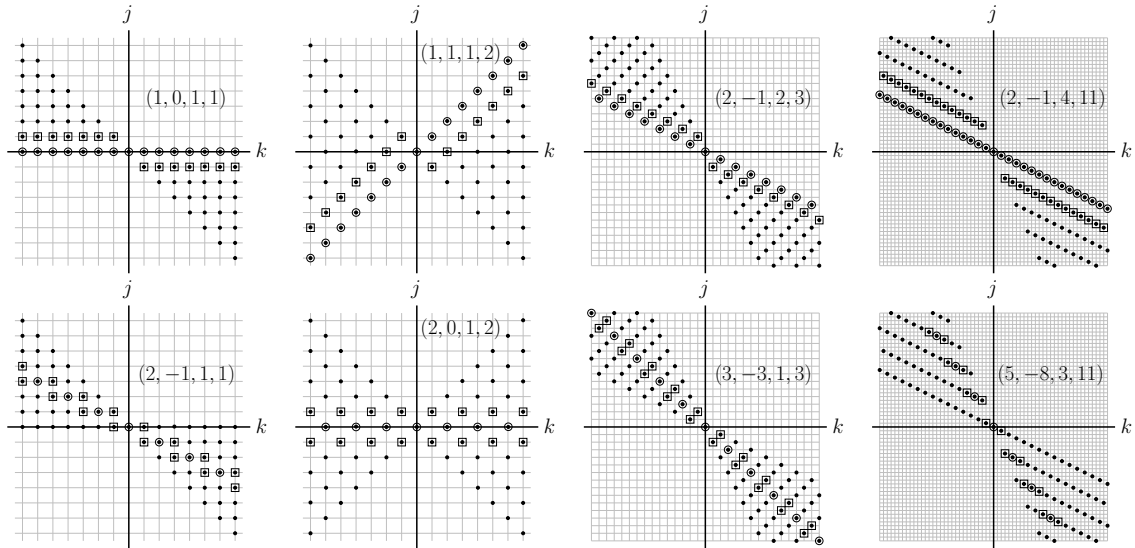


Figure 9: Each pair (aligned vertically) corresponds to a path of non-trivial solutions connecting two traveling waves. Solid dots represent the non-zero entries c_{kj} in (53) of the exact solutions along this path; open circles represent a traveling wave; and open squares represent the non-zero entries d_{kj} in the linearization about the traveling wave.

that $c_{kj} \neq 0$ for solutions on the paths

$$\begin{aligned} (1, 0, 1, 1) &\longleftrightarrow (2, -1, 1, 1), & (2, -1, 2, 3) &\longleftrightarrow (3, -3, 1, 3), \\ (1, 1, 1, 2) &\longleftrightarrow (2, 0, 1, 2), & (2, -1, 4, 11) &\longleftrightarrow (5, -8, 3, 11). \end{aligned} \quad (54)$$

All solutions on a given path have the same lattice pattern (of solid dots), but different paths have different patterns. One may show that if $u(x, t)$ is of the form (53) and

$$\frac{k}{2} \sum_{l,p} c_{lp} c_{k-l,j-p} = \left(k|k| + \frac{2\pi}{T} j \right) c_{kj}, \quad (k > 0, j \in \mathbb{Z}), \quad (55)$$

then $u(x, t)$ satisfies the Benjamin-Ono equation, $uu_x = Hu_{xx} - u_t$. The traveling waves at each end of the path have fewer non-zero entries, namely

$$\tilde{c}_{kj} = \left\{ \begin{array}{ll} N\alpha + \frac{2\pi\nu}{NT} & k = j = 0, \\ 2N\beta^{|k|/N} & k \in N\mathbb{Z} \setminus \{0\}, j = \frac{\nu k}{N} \\ 0 & \text{otherwise.} \end{array} \right\}, \quad \left(\alpha = \frac{1 - 3\beta^2}{1 - \beta^2} \right). \quad (56)$$

Here a tilde is used to indicate a solution about which we linearize. Substitution of $c_{kj} = \tilde{c}_{kj} + \varepsilon d_{kj}$ into (55) and matching terms of order ε leads to an eigenvalue problem with solution

$$d_{kj} = \left\{ \begin{array}{ll} \hat{z}_{N,n,k}, & k \in k_{N,n} + N\mathbb{Z}, \quad j = \frac{k\nu - m}{N}, \\ \hat{z}_{N,n,-k}, & k \in -k_{N,n} + N\mathbb{Z}, \quad j = \frac{k\nu + m}{N}, \\ 0 & \text{otherwise,} \end{array} \right. \quad (57)$$

with $\hat{z}_{N,n,k}$ as in (15). The non-zero coefficients d_{kj} in this linearization are represented by open squares in Figure 9. Recall from (15) that if $n \geq N$ and $k \leq n - N$ then $\hat{z}_{N,n,k} = 0$, but if $n < N$, the non-zero entries of $\hat{z}_{N,n,k}$ continue in both directions (with k approaching $+\infty$ or $-\infty$). This is why the rows of open squares terminate in graphs (a), (c), (e) and (g) rather than continuing past the origin as in graphs (b), (d), (f) and (h).

4.2 Elementary Symmetric Functions

It is interesting that the lattice patterns that arise for the exact solutions (beyond the linearization) contain only positive integer combinations of the lattice points of the linearization and of the traveling wave (treating the left and right half-planes separately). Somehow the double convolution in (55) leads to exact cancellation at all other lattice sites! This suggests that the c_{kj} have a highly regular structure that generalizes the simple power law decay rate of the Fourier coefficients $\hat{u}_{N,\beta,k}$ of the N -hump traveling wave.

The first step to understand this is to realize that there is a close connection between the trajectories of the Fourier coefficients and the trajectories of the elementary symmetric functions of the particles β_1, \dots, β_N in (2) above. Specifically, because the Fourier coefficients of $u_\beta(x)$ in (4) are of the form $2\beta^k$ for $k \geq 1$, we have

$$\beta_1^k(t) + \dots + \beta_N^k(t) = \frac{1}{2}c_k(t), \quad \left(k \geq 1, c_k(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x, t) e^{-ikx} dx\right). \quad (58)$$

Next we define the elementary symmetric functions σ_j via

$$\sigma_0 = 1, \quad \sigma_j = \sum_{l_1 < \dots < l_j} \beta_{l_1} \dots \beta_{l_j}, \quad (j = 1, \dots, N) \quad (59)$$

so that

$$P(z) := \prod_{l=1}^N (z - \beta_l) = \sum_{j=0}^N (-1)^j \sigma_j z^{N-j}. \quad (60)$$

It is well known [32] that the companion matrix Σ of P has the Jordan canonical form

$$\Sigma = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \pm\sigma_N & \dots & -\sigma_2 & \sigma_1 \end{pmatrix}, \quad V^{-1}\Sigma V = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}, \quad J_r = \begin{pmatrix} \beta_{l(r)} & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \beta_{l(r)} \end{pmatrix},$$

where $l : \{1, \dots, m\} \rightarrow \{1, \dots, N\}$ is an enumeration of the *distinct* roots of $P(z) = 0$ and the size of the Jordan block J_r is equal to the multiplicity of $\beta_{l(r)}$. As a result, the trace of powers of Σ will give the power sums of the β_l , and hence the Fourier coefficients:

$$c_k = 2 \operatorname{tr}(\Sigma^k), \quad (k \geq 1). \quad (61)$$

Thus, if the elementary symmetric functions are finite sums of circular orbits, then the Fourier coefficients will be as well, and we expect higher Fourier modes to involve more terms, in accordance with our findings above.

After extensive experimentation with data fitting on the numerical simulations described in Section 3, we have come to the following conclusion, which is proved in [31]:

CONJECTURE 3 *Up to a phase shift in space and time, all solutions on the path connecting (N, ν, n, m) to (N', ν', n', m') with $N' < N$ are of the form*

$$u(x, t) = \alpha_0 + \sum_{l=1}^N u_{\beta_l(t)}(x), \quad \hat{u}_{\beta, k} = \begin{cases} 2\bar{\beta}^{|k|}, & k < 0, \\ 0, & k = 0, \\ 2\beta^k, & k > 0, \end{cases} \quad (62)$$

where $\beta_1(t), \dots, \beta_N(t)$ are the zeros of the polynomial

$$P(z) = z^N + Ae^{-i\nu'\omega t}z^{N-N'} + Be^{-i(\nu-\nu')\omega t}z^{N'} + Ce^{-i\nu\omega t}. \quad (63)$$

Here A, B, C and $\omega = 2\pi/T$ are real constants depending on the mean α_0 and a bifurcation parameter such as $\rho = C$ or $\rho = \text{Re}\{c_k(0)\}$ for some integer k .

When N is even and $N' = N/2$, the two middle terms in (63) coalesce and $\sigma_{N'} = (-1)^{N'} [Ae^{-i\nu'\omega t} + Be^{-i(\nu-\nu')\omega t}]$ is a sum of two circular orbits. In all other cases, each elementary symmetric function executes circular orbits around the origin, or remains constant in time. Note that at most 4 of the $N + 1$ elementary symmetric functions are non-zero.

The choice $\rho = \text{Re}\{c_k(0)\}$ with $k = N - N'$ is a natural choice for the bifurcation parameter in numerical experiments as the linearization about each traveling wave involves this Fourier mode, i.e. $\hat{z}_{N', N-1, k} \neq 0$ and $\hat{z}_{N, N-N', k} \neq 0$ in (15). The drawback is that ρ may have a turning point in the transition from the N' -hump traveling wave to the N -hump traveling wave while another choice of k may yield a ρ that varies monotonically. In the analysis below we use $\rho = C$ as we are able to solve for A and B in terms of C .

Once the mapping (49) from (N, ν, n, m) to (N', ν', n', m') is known, we can choose N, ν, N' and ν' independently, subject to the conditions

$$N' < N, \quad \nu' > \frac{N'}{N}\nu. \quad (64)$$

The first condition is merely a labeling convention while the second is an actual restriction on which traveling waves are connected together by a path of non-trivial solutions. The formulas of Conjecture 2 then imply that

$$m = m' = N\nu' - N'\nu > 0, \quad n = N - N', \quad n' = N - 1. \quad (65)$$

In terms of $N, \nu, N', \nu', \alpha = \frac{1-3|\beta|^2}{1-|\beta|^2}$ and $\alpha' = \frac{1-3|\beta'|^2}{1-|\beta'|^2}$, (29) and (31) give

$$\begin{aligned} T &= \frac{2\pi(N\nu' - N'\nu)}{N'(N - N')N}, & \alpha_0 &= \alpha_0^* - (1 - \alpha)N, & \alpha_0^* &:= \frac{N^2\nu' - (N')^2\nu}{N\nu' - N'\nu} \\ T' &= \frac{2\pi(N\nu' - N'\nu)}{N'(N - N')[N + (1 - \alpha')N']}, & \alpha'_0 &= \alpha_0^* - \frac{\nu' - \nu}{N\nu' - N'\nu}(N')^2(1 - \alpha'). \end{aligned} \quad (66)$$

We note that the two traveling waves reduce to the same constant function when $\beta \rightarrow 0$ and $\beta' \rightarrow 0$, which is further evidence that a sheet of non-trivial solutions connects these two families of traveling waves. We can think of the non-trivial solutions of the form (63) as a manifold with boundary that lies between the two families of traveling waves obtained by setting $B = C = 0$ or $A = B = 0$. Actually, since A, B and C are allowed to be negative,

there are two manifolds with boundary connecting the traveling waves; taken together, they form a manifold without boundary that contains the traveling waves as submanifolds.

To make these statements rigorous, one could try to substitute the ansatz (63) for σ_j into (61) and solve the lattice equation (55) for A , B and ω in terms of C and α_0 . We have not succeeded in doing this analytically, but our numerical solutions do yield symmetric functions that satisfy these equations to 12 decimal places. A simpler approach is to express the solution directly in terms of the elementary symmetric functions via

$$\begin{aligned} u(x, t) &= \alpha_0 + \sum_{l=1}^N u_{\beta_l(t)}(x) = \alpha_0 + \sum_{l=1}^N 4 \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \beta_l(t)^k e^{ikx} \right\} \\ &= \alpha_0 + \sum_{l=1}^N 4 \operatorname{Re} \left\{ \frac{z}{z - \beta_l(t)} - 1 \right\} = \alpha_0 + 4 \operatorname{Re} \left\{ \frac{z \partial_z P(z)}{P(z)} - N \right\}, \quad (z = e^{-ix}), \end{aligned} \quad (67)$$

and then to derive algebraic expressions relating A , B , C , α_0 , ω , N , N' , ν and ν' by substituting (67) into the Benjamin-Ono equation (1). To this end, we include the time dependence of P in the notation and write (67) in the form

$$u(x, t) = \alpha_0 + 2 \left(\frac{i \partial_x g}{g} - N \right) + 2 \left(\frac{-i \partial_x h}{h} - N \right), \quad (68)$$

where

$$g(x, t) = P(e^{-ix}, e^{-i\omega t}), \quad h(x, t) = \overline{g(x, t)}, \quad (69)$$

$$P(z, \lambda) = z^N + A \lambda^{\nu'} z^{N-N'} + B \lambda^{\nu-\nu'} z^{N'} + C \lambda^{\nu}. \quad (70)$$

Note that P is a polynomial in z and a Laurent polynomial in λ (as ν and ν' may be negative). We may assume $\omega > 0$; if not, we can change the sign of ω without changing the solution by replacing (A, B, ν, ν', N') by $(B, A, -\nu, \nu' - \nu, N - N')$. Assuming the roots $\beta_l(t)$ of $P(z, e^{-i\omega t})$ remain inside the unit disk Δ of the complex plane, we have

$$\left(\frac{i \partial_x g}{g} - N \right) = \sum_{l=1}^N \sum_{k=1}^{\infty} \beta_l(t)^k e^{ikx} \Rightarrow Hu = 2 \left(\frac{\partial_x g}{g} + Ni \right) + 2 \left(\frac{\partial_x h}{h} - Ni \right). \quad (71)$$

Using (68) and $\partial_t \left(\frac{\partial_x g}{g} \right) = \partial_x \left(\frac{\partial_t g}{g} \right)$, the equation $\frac{1}{2} (u_t - Hu_{xx} + uu_x) = 0$ becomes

$$\partial_x \left[i \left(\frac{\partial_t g}{g} - \frac{\partial_t h}{h} \right) - \partial_x \left(\frac{\partial_x g}{g} + \frac{\partial_x h}{h} \right) + \frac{1}{4} \left((\alpha_0 - 4N) + 2i \left(\frac{\partial_x g}{g} - \frac{\partial_x h}{h} \right) \right)^2 \right] = 0. \quad (72)$$

The expression in brackets must be a constant, which we denote by γ . We now write

$$P_{jk} = (z \partial_z)^j (\lambda \partial_\lambda)^k P(z, \lambda) \Big|_{\substack{z=e^{-ix} \\ \lambda=e^{-i\omega t}}} \quad (73)$$

so that e.g. $\partial_t g = -i\omega P_{01}$ and $\partial_x h = i\bar{P}_{10}$. Equation (72) then becomes

$$\begin{aligned} \gamma P_{00}\bar{P}_{00} + \bar{P}_{00}[P_{20} + \omega P_{01} + (\alpha_0 - 4N)P_{10}] \\ + P_{00}[\bar{P}_{20} + \omega\bar{P}_{01} + (\alpha_0 - 4N)\bar{P}_{10}] + 2P_{10}\bar{P}_{10} = 0, \end{aligned} \quad (74)$$

where we have absorbed $\frac{1}{4}(\alpha_0 - 4N)^2$ into γ . This equation may be written

$$e_1 \llbracket z^N \lambda^{-\nu} \rrbracket + e_2 \llbracket z^{N-2N'} \lambda^{2\nu'-\nu} \rrbracket + e_3 \llbracket z^{N-N'} \lambda^{\nu'-\nu} \rrbracket + e_4 \llbracket z^{N'} \lambda^{-\nu'} \rrbracket + e_5 = 0,$$

where $\llbracket a \rrbracket = a + \bar{a} = 2\operatorname{Re}\{a\}$,

$$e_1 = [\gamma + \nu\omega + N^2 + (\alpha_0 - 4N)N]C, \quad e_2 = [\gamma + \nu\omega + N^2 + (\alpha_0 - 4N)N]AB,$$

and, after setting $\gamma = (3N - \alpha_0)N - \nu\omega$ to achieve $e_1 = e_2 = 0$,

$$e_3 = [(N')^2 - 2NN' + N'\alpha_0 - \nu'\omega]B + [(N')^2 + 2NN' - N'\alpha_0 + \nu'\omega]AC = 0, \quad (75)$$

$$\begin{aligned} e_4 = [3N^2 - 4NN' + (N')^2 - (N - N')\alpha_0 + (\nu - \nu')\omega]BC \\ - [N^2 - (N')^2 - (N - N')\alpha_0 + (\nu - \nu')\omega]A = 0, \end{aligned} \quad (76)$$

$$\begin{aligned} e_5 = (N\alpha_0 - \nu\omega - N^2) + [(2N' - N)\alpha_0 + (\nu - 2\nu')\omega + 3N^2 - 8NN' + 4(N')^2]B^2 \\ + [(N - 2N')\alpha_0 + 4(N')^2 - N^2 + (2\nu' - \nu)\omega]A^2 + [(3N - \alpha_0)N + \nu\omega]C^2 = 0. \end{aligned} \quad (77)$$

These three equations can be solved for A , B and ω in terms of C , α_0 , N , N' , ν and ν' . The result will yield a solution of (1) if the roots of $P(\cdot, e^{-i\omega t})$ remain inside the unit disk for all $t \in \mathbb{R}$. This will be proved elsewhere [31].

4.3 Three Types of Reconnection

We now wish to explain why following a path of non-trivial solutions with the mean α_0 held fixed sometimes leads to re-connection with a different traveling wave and sometimes leads to blow-up. We consider three cases in solving the algebraic equations (75)–(77):

Case 1: ($\nu' = 0$) In this case, we are dealing with the path connecting an N' -hump stationary solution to an N -hump traveling wave that moves left with speed index $\nu < 0$, by (64). We substitute $B = AC/E$ and solve (75) for E , which gives

$$E = \frac{2N - N' - \alpha_0}{2N + N' - \alpha_0}, \quad B = \frac{AC}{E}. \quad (78)$$

Then (76) can be used to solve for ω , which yields

$$\omega = \frac{(2N - N' - \alpha_0)(N + N' - \alpha_0) - (2N + N' - \alpha_0)(3N - N' - \alpha_0)C^2}{(2N - N' - \alpha_0) - (2N + N' - \alpha_0)C^2} \left(\frac{N - N'}{-\nu} \right). \quad (79)$$

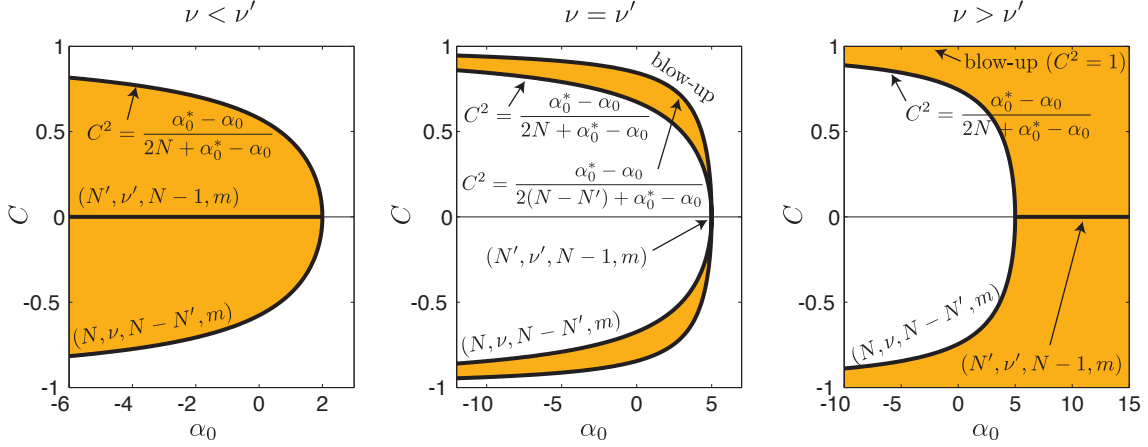


Figure 10: The shape of the feasible set (shaded region) for solutions on the path from (N', ν', n', m') to (N, ν, n, m) depends on the sign of $\nu - \nu'$. Here $\alpha_0^* = \frac{N^2 \nu' - (N')^2 \nu}{N \nu' - N' \nu}$ is the value of the mean for the constant solution that both traveling waves reduce to when $\beta \rightarrow 0$. On the α_0 axis when $\nu \neq \nu'$, we have $C = 0$, $B = 0$ and $A^2 = \frac{m(\alpha_0^* - \alpha_0)}{2(N')^2(\nu' - \nu) + m(\alpha_0^* - \alpha_0)}$.

Once ω and E are known, (77) can be solved to obtain

$$A^2 = \frac{(2N - N' - \alpha_0)^2 [(2N - N' - \alpha_0) - (4N - N' - \alpha_0)C^2] [(N' - \alpha_0) - (2N + N' - \alpha_0)C^2]}{[(2N - N' - \alpha_0)^2 - (2N + N' - \alpha_0)(4N - 3N' - \alpha_0)C^2] [(2N - N' - \alpha_0)(3N' - \alpha_0) - (2N + N' - \alpha_0)^2 C^2]}. \quad (80)$$

This case always yields a re-connection when the mean is held constant. From (66), we see that the stationary and traveling waves at each end of the path require $\alpha_0 \leq \alpha_0^* = N'$. Holding $\alpha_0 < N'$ fixed and varying C^2 from 0 (stationary) to $\frac{N' - \alpha_0}{2N + N' - \alpha_0}$ (traveling), we find that A^2 decreases from $\frac{N' - \alpha_0}{3N' - \alpha_0}$ to 0 while the period $T = \frac{2\pi}{\omega}$ increases from $\frac{2\pi(-\nu)}{N^2 - (N')^2 - (N - N')\alpha_0}$ to $\frac{2\pi(-\nu)}{N(N - N')}$. This path could be represented by a vertical line through the shaded region in the first panel of Figure 10 (which also represents one possibility in Case 3 below). We may choose the signs of A and C arbitrarily as long as B is chosen to have the sign of their product (since E is positive). Although the heavy lines do have an interpretation as a bifurcation diagram from constant solutions to stationary and traveling waves, this figure is primarily a plot of feasibility versus C and α_0 jointly.

Case 2: ($\nu = \nu'$) In this case, $\nu > 0$ due to (64). The algebraic structure of the equations is similar to Case 1 with $N' \leftrightarrow N - N'$ and $A \leftrightarrow B$ due to the form of P in (70); however, the feasible set for C and α_0 is completely different; see Figure 10. This time we use (76) instead of (75) to determine E :

$$E = \frac{N + N' - \alpha_0}{3N - N' - \alpha_0}, \quad A = \frac{BC}{E}. \quad (81)$$

We then use (75) to compute ω :

$$\omega = \frac{(3N - N' - \alpha_0)(2N + N' - \alpha_0)C^2 - (N + N' - \alpha_0)(2N - N' - \alpha_0)}{(N + N' - \alpha_0) - (3N - N' - \alpha_0)C^2} \left(\frac{N'}{\nu} \right). \quad (82)$$

Substituting these into (77), we can solve for B :

$$B^2 = \frac{(N + N' - \alpha_0)^2 [(N + N' - \alpha_0) - (3N + N' - \alpha_0)C^2] [(N - N' - \alpha_0) - (3N - N' - \alpha_0)C^2]}{[(N + N' - \alpha_0)^2 - (3N - N' - \alpha_0)(N + 3N' - \alpha_0)C^2] [(N + N' - \alpha_0)(3N - 3N' - \alpha_0) - (3N - N' - \alpha_0)^2 C^2]}. \quad (83)$$

This case always leads to blow-up unless α_0 and C are simultaneously varied to approach $(\alpha_0^*, 0)$ while remaining in the shaded region shown in the middle panel of Figure 10. If we hold $\alpha_0 < \alpha_0^* = N + N'$ fixed, the bifurcation from the N -hump traveling solution occurs at $C^2 = \frac{N + N' - \alpha_0}{3N + N' - \alpha_0}$. This is not the same value as in Case 1 with N' replaced by $N - N'$ because ν is now positive, so the range of C^2 that worked in Case 1 leads to negative values of ω now, and is properly accounted (with $\omega > 0$) by a different path. Instead, in the current case, the other bracketed term of the numerator of (83) is zero at the bifurcation, and as we *increase* C^2 over the range

$$\frac{N + N' - \alpha_0}{3N + N' - \alpha_0} \leq C^2 < \frac{N + N' - \alpha_0}{3N - N' - \alpha_0}, \quad (84)$$

B^2 increases from 0 to $\frac{N + N' - \alpha_0}{3N - N' - \alpha_0}$, A^2 increases from 0 to 1, $T = \frac{2\pi}{\omega}$ decreases from $\frac{2\pi\nu}{NN'}$ to 0, and the L^2 norm of the solution blows up due to the fact that N' of the roots of $z \mapsto P(z, \lambda)$ in (70) are on the unit circle in the limiting case: $P(\sqrt[N']{-\lambda\nu}, \lambda) = 0$.

The inequalities (84) give the feasible set of values of the bifurcation parameters C and α_0 for this case. In order to approach the bifurcation (N', ν', n', m') , we must simultaneously increase α_0 to $N + N'$ while decreasing C^2 so that it remains between the two bounding curves in (84). More systematically, let us introduce a new parameter F such that

$$C^2 = \frac{N + N' - \alpha_0}{3N + N' - 2N'F^2 - \alpha_0}. \quad (85)$$

Then for fixed $\alpha_0 < N + N'$, the N -hump traveling wave occurs at $F = 0$ and the solution blows up as F increases to 1; meanwhile, for fixed $F \in [0, 1)$, as α_0 increases to $N + N'$ the non-trivial solution approaches the N' -hump traveling wave with $B \rightarrow 0$, $C \rightarrow 0$, $E \rightarrow 0$ and $A^2 \rightarrow \frac{(N - N')F^2}{(N - 2N')F^2 + N'}$, showing that all values of A between 0 and 1 are achievable. Note that when $B = C = 0$, we can interpret $|A|$ as the decay parameter $|\beta|$ of the N' -hump traveling wave.

Case 3: ($\nu' \neq 0$ and $\nu' \neq \nu$) As in Case 1, we define $E = AC/B$, but this time we use (75) to solve for ω instead of E :

$$\omega = \frac{(2N - N' - \alpha_0) - (2N + N' - \alpha_0)E}{1 - E} \left(\frac{N'}{-\nu'} \right), \quad B = \frac{AC}{E}. \quad (86)$$

This formula for ω causes (76) to become quadratic in E and independent of A , and causes (77) to become linear in A^2 and cubic in E . Only one of the two roots E of (76) corresponds to the path from $(N', \nu', N-1, m)$ to $(N, \nu, N-N', m)$; we discard the other root as it yields solutions on the path from $(N-N', \nu'-\nu, N-1, m)$ to $(N, -\nu, N', m)$ with negative values of ω . The correct root is the one for which the numerator in the quadratic formula does not cancel as $C \rightarrow 0$. The sign of the square root in the quadratic formula (and the side of the curve $C^2 = \frac{\alpha_0^* - \alpha_0}{2N + \alpha_0^* - \alpha_0}$ that is feasible) depends on whether $\nu < \nu'$ or $\nu > \nu'$. We omit the general formulas for A^2 and E as they are very complicated, and instead present a typical example. (Another example is given in Section 5 below.)

EXAMPLE 4 Consider the three-particle solutions on the path $e : (2, -1, 2, 3) \leftrightarrow (3, -3, 1, 3)$ in Figures 2 and 4. Since $\nu' = -1$ while $\nu = -3$, the feasible region is of the first type shown in Figure 10, i.e. we do not need to vary the mean in order to reconnect with a traveling wave on the other side of the path. Suppose $\alpha_0 < \alpha_0^* = 1$. Then proceeding as described in Case 3 above, we find that $u(x, t)$ is of the form (67) with

$$P(z) = z^3 + Ae^{i\omega t}z + Be^{2i\omega t}z^2 + Ce^{3i\omega t}, \quad (87)$$

where

$$\begin{aligned} \omega &= \frac{2[(4 - \alpha_0) - (8 - \alpha_0)E]}{1 - E}, \quad B = \frac{AC}{E}, \\ A^2 &= \frac{3E^2[(5 - \alpha_0) + (1 + \alpha_0)C^2 - (13 - \alpha_0)E + (7 - \alpha_0)EC^2]}{(13 - 3\alpha_0)C^2 - 3(7 - \alpha_0)EC^2 - 3(5 - \alpha_0)E^2 + (23 - 3\alpha_0)E^3}, \\ E &= \frac{(11 - 3\alpha_0) + (25 - 3\alpha_0)C^2 + \sqrt{[(11 - 3\alpha_0) - (25 - 3\alpha_0)C^2]^2 + 128C^2}}{6(9 - \alpha_0)}. \end{aligned} \quad (88)$$

The transition from the two- to three-hump traveling wave occurs as we vary the bifurcation parameter C from 0 to $\sqrt{\frac{1-\alpha_0}{7-\alpha_0}}$, which causes E to increase from $\frac{11-3\alpha_0}{27-3\alpha_0}$ to $\frac{3-\alpha_0}{7-\alpha_0}$ and A to decrease from $\sqrt{\frac{3-3\alpha_0}{19-3\alpha_0}}$ to 0. B is zero at both ends of the path.

The trajectories $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$ for $\alpha_0 = 0.544375$ and four choices of C are shown in Figure 11. Note that the bifurcation from the two-hump traveling wave causes a new particle to nucleate at the origin. As C increases, its trajectory grows in amplitude until it joins up with the orbits of the outer particles. There is a critical value of C at which the particles collide and the solution of the ODE (3) ceases to exist for all time; nevertheless, the representation of u in terms of P in (67) remains well-behaved and does satisfy (1) for all time. Thus, a change in topology of the orbits does not manifest itself as a singularity in the solution of the PDE. As C increases further, the three orbits become nearly circular and

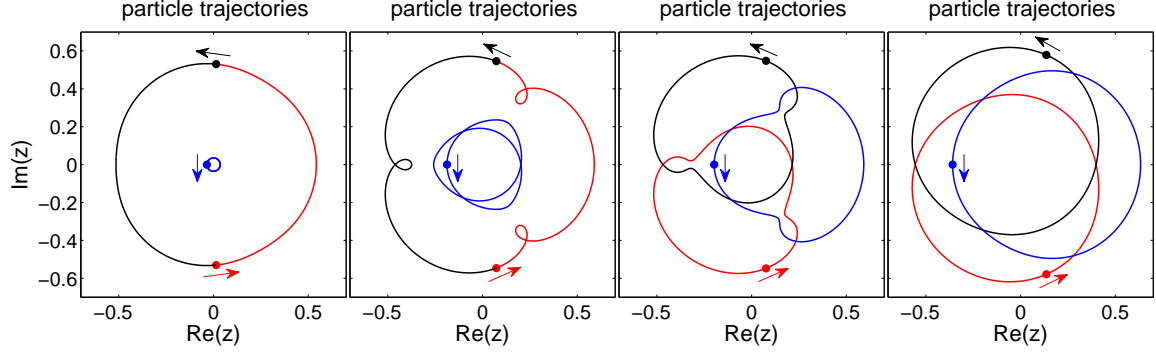


Figure 11: Trajectories $\beta_l(t)$ for four solutions on the path $(2, -1, 2, 3) \leftrightarrow (3, -3, 1, 3)$ with mean $\alpha_0 = 0.544375$. The markers give the position of the β_l at $t = 0$. The value of C in (87) is, from left to right: 0.00988, 0.05690, 0.06038 and 0.1268. The value of C in Figure 4 is $C = 0.1949$; the solution becomes a 3-hump traveling wave at $C = 0.2657$.

eventually coalesce into a single circular orbit (with $\nu = -3$) at the three-hump traveling wave. The “braided” effect of the solution shown in Figure 4 is recognizable for $C \geq 0.1$ or so for this value of the mean.

5 Interior Bifurcations

We conclude this work by mentioning that our numerical method for following paths of non-trivial solutions from one traveling wave to another occasionally wanders off course, following an interior bifurcation rather than reaching the traveling wave on the other side of the original path. These interior bifurcations lead to new paths of non-trivial solutions that are more complicated than those on the original path. For example, on the path

$$(1, 1, 1, 2) \longleftrightarrow (2, 0, 1, 2), \quad (89)$$

Conjecture 3 suggests that the exact solution is a two-particle solution with elementary symmetric functions of the form

$$\sigma_1(t) = -(Ae^{-i\omega t} + Be^{i\omega t}), \quad \sigma_2(t) = C. \quad (90)$$

Following the procedure described in Case 3 of the previous section shows that this is indeed the case if $\alpha_0 < \alpha_0^* = 2$, $C^2 \leq \frac{2-\alpha_0}{6-\alpha_0}$, and we define

$$A^2 = \frac{[(2-\alpha_0) - (6-\alpha_0)C^2](1-E)E^2}{[(5-\alpha_0)E - (3-\alpha_0)](E^2 - C^2)}, \quad B = \frac{AC}{E}, \quad \omega = \frac{(5-\alpha_0)E - (3-\alpha_0)}{1-E},$$

$$E = \frac{(3-\alpha_0) + (5-\alpha_0)C^2 + \sqrt{[(3-\alpha_0) - (5-\alpha_0)C^2]^2 - 4C^2}}{2(4-\alpha_0)}. \quad (91)$$

In Figure 12, we show the bifurcation diagram for the transition from the one-hump right-traveling wave (labeled P) to the two-hump stationary solution (labeled Q). This diagram was computed numerically before we had any idea that exact solutions for this problem exist; therefore, we used the real part of the first Fourier mode at $t = 0$ for the bifurcation parameter rather than C . We can obtain the same curves analytically as follows. The upper curve from P to Q (containing A1-A5) can be plotted parametrically by varying C in (91) from 0 to $-\sqrt{\frac{2-\alpha_0}{6-\alpha_0}}$ holding $\alpha_0 = 0.544375$ fixed, taking the positive square root for A , and plotting $-2(A+B)$ versus $T = \frac{2\pi}{\omega}$. The lower curve from P to Q is obtained by varying C from 0 to $\sqrt{\frac{2-\alpha_0}{6-\alpha_0}}$ instead. In either case, as C^2 increases from 0 to $\frac{2-\alpha_0}{6-\alpha_0}$, E decreases from $\frac{3-\alpha_0}{4-\alpha_0}$ to $\frac{4-\alpha_0}{6-\alpha_0}$, A^2 decreases from $\frac{2-\alpha_0}{3-\alpha_0}$ to 0, $T = \frac{2\pi}{\omega}$ increases from $\frac{2\pi}{3-\alpha_0}$ to 2π , and B^2 increases from 0 to a maximum value before decreasing monotonically back to zero.

As illustrated in Figure 12, solutions such as A1-A5 on the upper path have $\sigma_1(t)$ executing elliptical, clockwise orbits that start out circular at the one-hump traveling wave but become more eccentric and collapse to a point as we progress toward the two-hump stationary solution Q. Meanwhile, $\sigma_2(t)$ remains constant in time, nucleating from the origin at the one-hump traveling wave and terminating with $\sigma_2 \equiv -\sqrt{\frac{2-\alpha_0}{6-\alpha_0}}$ at the two-hump stationary solution. On the lower path, the major axis of the orbit of σ_1 is horizontal rather than vertical and σ_2 moves right rather than left as we move from P to Q.

When computing these paths from P to Q, we encountered two interior bifurcations. In the bifurcation labeled B6 in Figure 12, an additional elementary symmetric function nucleates at the origin and the trajectories of σ_1 and σ_2 become more complicated. Through data fitting, we find that

$$\sigma_1(t) = -(Ae^{-i\omega t} + Be^{i\omega t} + C_1e^{3i\omega t}), \quad (92)$$

$$\sigma_2(t) = C + C_2e^{2i\omega t} + C_3e^{4i\omega t}, \quad (93)$$

$$\sigma_3(t) = -C_4e^{3i\omega t}, \quad (94)$$

where the new coefficients C_j are all real parameters. We have not attempted to derive algebraic relationships among these parameters to obtain exact solutions. These trajectories are shown in Figure 13 for the solutions labeled B1-B13 in the bifurcation diagram. The

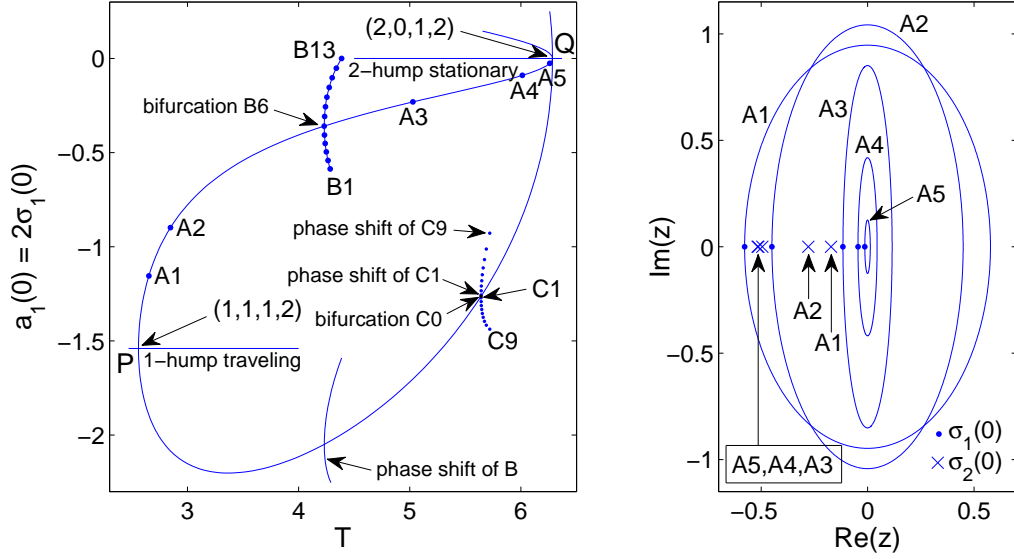


Figure 12: *Left*: Bifurcation diagram showing several interior bifurcations on the path $(1, 1, 1, 2) \rightarrow (2, 0, 1, 2)$. *Right*: Trajectories of the elementary symmetric functions $\sigma_1(t)$, which have elliptical, clockwise orbits, and $\sigma_2(t)$, which remain stationary in time, for the solutions labeled A1-A5 in the bifurcation diagram.

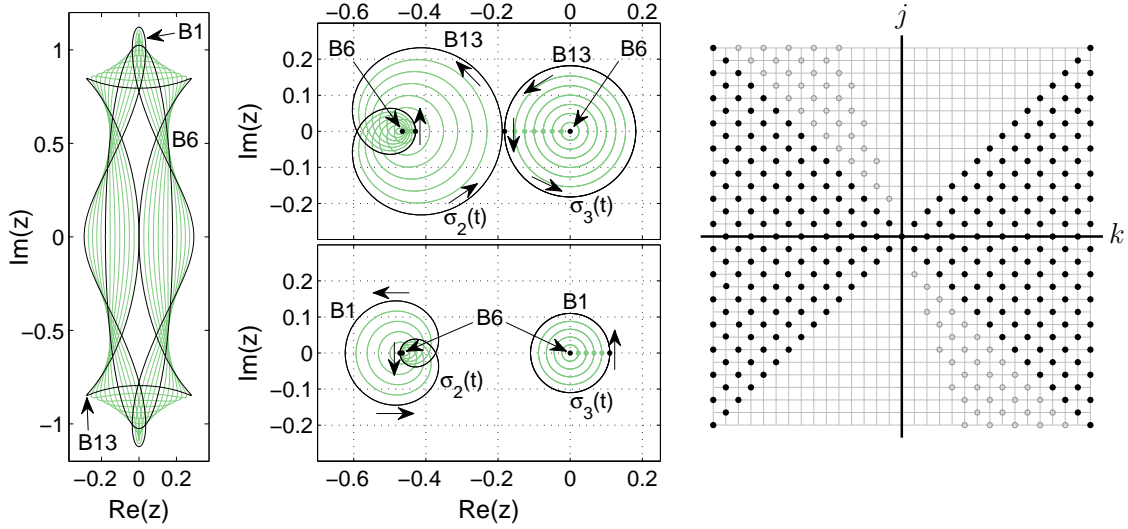


Figure 13: *Left*: Trajectories of $\sigma_1(t)$ for solutions labeled B1-B13 in Figure 12. *Center*: Trajectories of $\sigma_2(t)$ and $\sigma_3(t)$. Since B6 is on the original path from P to Q, $\sigma_2(t)$ is constant and $\sigma_3(t) \equiv 0$ for this solution. *Right*: The interior bifurcation causes additional lattice coefficients c_{kj} to become non-zero; grey circles represent the new terms.

additional term in (92) causes the elliptical orbit of $\sigma_1(t)$ to deform by bulging out in the vertical and horizontal directions while pulling in along the diagonal directions (or vice versa, depending on which direction we follow the bifurcation). Meanwhile, $\sigma_2(t)$ ceases to be constant and $\sigma_3(t)$ ceases to be zero. To avoid clutter, we plotted the trajectories $\sigma_2(t)$ and $\sigma_3(t)$ for B1-B6 separately from B6-B13, illustrating the effect of following the bifurcation in one direction or the other. The additional terms in (92)–(94) cause the lattice pattern of non-zero entries $c_{kj} = \frac{1}{T} \int_0^T c_k(t) e^{ij\omega t} dt$ to become more complicated, where we recall that in this case,

$$c_k(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x, t) e^{-ikx} dx = 2 \operatorname{tr} \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sigma_3(t) & -\sigma_2(t) & \sigma_1(t) \end{pmatrix}^k \right].$$

The solid dots in Figure 13 represent the non-zero entries of solutions on the original path from P to Q while grey circles show the additional terms that are non-zero after the bifurcation at B6. Although this bifurcation causes some of the unoccupied lattice sites to be filled in, the new lattice pattern is rather similar to the original pattern and maintains its checkerboard structure. Also, this bifurcation leads to symmetric perturbations of the Fourier mode trajectories, and is also present (in a phase shifted form) along the lower path from P to Q.

In the bifurcation labeled C0 in Figure 12, the fill-in pattern of the lattice representation is much more complicated, and in fact the checkerboard structure of the non-zero coefficients c_{kj} is destroyed; see Figure 14. But actually, the elementary symmetric functions behave similarly to the previous case: By fitting our numerical data, we find that

$$\sigma_1(t) = -(Ae^{-i\omega t} + Be^{i\omega t} + C_1e^{4i\omega t}), \quad (95)$$

$$\sigma_2(t) = C + C_2e^{3i\omega t} + C_3e^{5i\omega t}, \quad (96)$$

$$\sigma_3(t) = -C_4e^{4i\omega t}, \quad (97)$$

so each of the new terms executes one additional loop per cycle of the periodic solution in comparison to the corresponding term in (92)–(94). This extra loop causes a star-shaped perturbation of the σ_1 ellipse instead of the rectangular and diamond shaped perturbations seen previously in Figure 13. As a result, this bifurcation is not present on the upper path from P to Q because the symmetry of the perturbation does not respect the 90 degree rotation of the orbit $\sigma_1(t)$ associated with the $\frac{\pi}{2}$ -spatial and $\frac{T}{4}$ -temporal phase shifts that relate solutions on the upper and lower paths from P to Q. To follow the bifurcation at C0 in the other direction, we can use the same numerical values for $A, B, C, C_1, C_2, C_3, C_4$ in (95)–(97) after changing the signs of the latter four parameters. This causes the trajectories

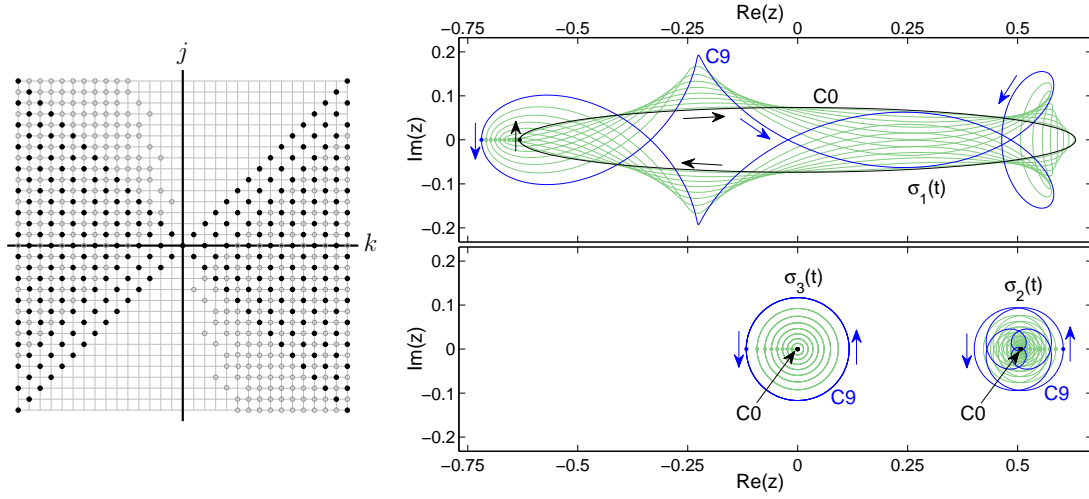


Figure 14: *Left:* This interior bifurcation causes more lattice coefficients to become non-zero than the interior bifurcation of Figure 13. *Right:* Trajectories of $\sigma_1(t)$, $\sigma_2(t)$, and $\sigma_3(t)$ for the solutions labeled C0-C9 in Figure 12. The long axis of the ellipse C0 is horizontal because we start from the bottom branch connecting P to Q in Figure 12.

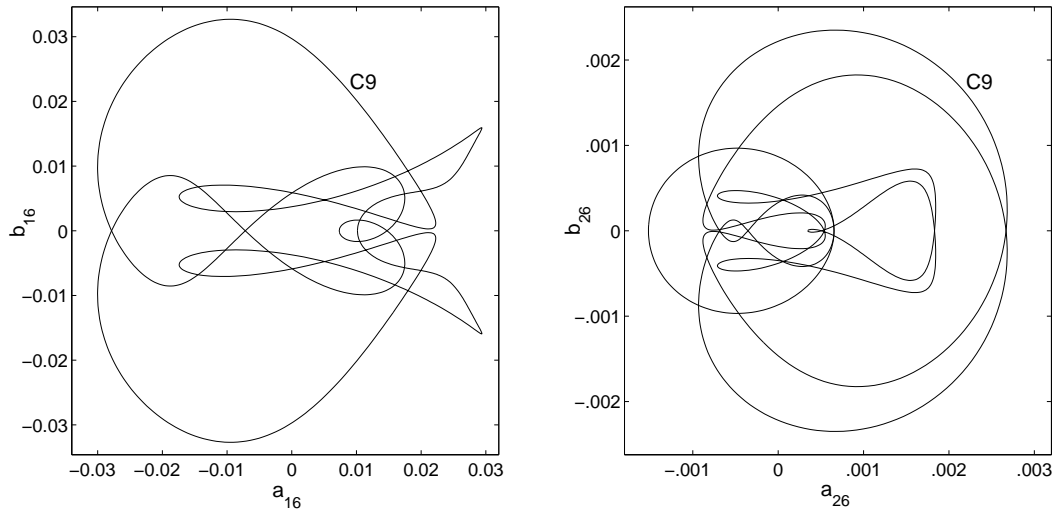


Figure 15: The trajectories of the Fourier modes become very complicated after the interior bifurcation occurs. Here we show the 16th and 26th Fourier modes $c_k(t) = a_k(t) + ib_k(t)$ over one period. It was clearly essential to use a high order (in fact spectrally accurate) numerical method to resolve these dynamics when computing time-periodic solutions.

of σ_1 in Figure 14 to be rotated 180° with a corresponding $\frac{T}{2}$ phase-shift in time so that the initial position $\sigma_1(0)$ remains on the left side of the figure. Meanwhile, the trajectory of $\sigma_2(t)$ experiences a $\frac{T}{2}$ phase-shift in time with no change in the location of the orbit, and $\sigma_3(t)$ starts on the opposite side of its circular trajectory about the origin.

In Figure 15, we show the orbits of the 16th and 26th Fourier modes for the solution labeled C9 in the bifurcation diagram of Figure 12. As the index of the Fourier mode increases, these trajectories become increasingly complicated (involving more non-zero terms c_{kj} in the lattice representation), but also decay exponentially so that the amplitude of the orbit is eventually smaller than can be resolved using floating point arithmetic. We emphasize that these trajectories were resolved to full machine precision by our general purpose numerical method for finding periodic solutions of non-linear PDE (without any knowledge of the solitonic structure of the solutions). Everything we learned about the form of the exact solutions came about from studying these numerical solutions, which was possible only because our numerical results are correct to 10-15 digits of accuracy.

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